15

THE ELEMENTS

OF

TRIGONOMETRY.

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BY

S. L. LONEY

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PREFACE.

THE following work is intended for the use of Students commencing Trigonometry. It consists of the easier portions of Part I. of my *Plane Trigonometry*, but a re-arrangement has been made in order that the Student may not meet with all the difficulties of the subject at the outset. Thus the Circular Measure of angles is not introduced until Chapter XIV, whilst the trigonometrical functions of angles greater than two right angles are first dealt with in Chapter XV. The Student's attention is, from the commencement, chiefly directed to what he would practically want for the Solution of Triangles.

In the logarithmic part of the book I have generally confined the work to four and five-figure logarithms; the latter give sufficient accuracy for nearly all practical purposes, and the former for most questions that arise. I have however included a certain number of questions involving seven-figure logarithms for practice in more accurate working.

In order to encourage the actual use of logarithmic tables, I have had printed at the end of the book 20 pages of four-figure logarithms. I hope these will

be found useful in solving the many questions in the book where the Student is left to look out the logarithms for himself. These tables are those used by the Cambridge Local Examinations Syndicate for their examinations, and I am much indebted to the Syndicate for kindly allowing me to print them. Had the size of the page used been sufficient, I should have preferred five-figure tables, but it seemed undesirable to sacrifice clearness of printing.

The answers have generally been obtained by the use of a five-figure table of logarithms. The four-figure tables will not always give exactly the same results.

As in my larger book I have prefixed a list of the principal formulæ which the Student should commit to memory. These more important formulæ are distinguished in the text by the use of thick type.

The number of examples being very large, a selection only should be attempted by the Student at a first reading. He should also, on his first reading, omit all articles marked with an asterisk.

For any corrections, or suggestions for improvement, I should be thankful.

S. L. LONEY.

ROYAL HOLLOWAY COLLEGE
(UNIVERSITY OF LONDON),
EGHAM, SURREY.
Nov. 10, 1904.

CONTENTS.

CHAPTER I.

1. TRIGONOMETRY is the name given to that branch of Mathematics which deals primarily with the relations between the sides and angles of a triangle.

2. In geometry angles are measured generally in terms of a right angle. This however is an inconvenient unit on account of its size. A right angle is therefore divided into 90 equal parts called **Degrees**. Each degree is divided into 60 equal parts called **Minutes**, and each minute into 60 equal parts called **Seconds**. This system of measurement is called the **Sexagesimal** system.

The symbols 1°, $1'$, and $1''$ are used to denote a degree, a minute, and a second respectively.

Thus, 60 Seconds ($60''$) make one Minute ($1'$),
60 Minutes ($60'$) „ „ Degree (1°),
and 90 Degrees (90°) „ „ Right Angle.

3. This system of measurement is always used in the practical applications of Trigonometry. It is not however very convenient on account of the multipliers 60 and 90. In a later chapter the student will meet with other systems of measurement. It is easy to express degrees etc. in terms of right angles and *vice versâ*.

Ex. 1. *Express* $63^\circ 14' 51''$ *in terms of a right angle.*

We have $$51'' = \frac{51'}{60} = \frac{17'}{20} = \cdot 85',$$

and $$14' 51'' = 14\cdot 85' = \frac{14\cdot 85^\circ}{60} = \cdot 2475^\circ.$$

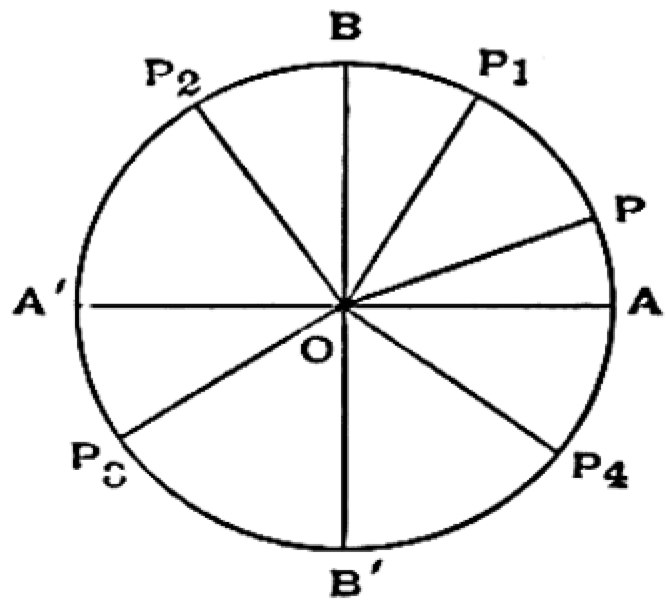
$$\therefore 63^\circ 14' 51'' = 63\cdot 2475^\circ = \frac{63\cdot 2475^\circ}{90} \text{ of a rt. } \angle$$
$$= \cdot 70275 \text{ rt. } \angle.$$

Ex. 2. *Express 3·467 rt. ∠ˢ in the Sexagesimal System.*

$3{\cdot}467 \text{ rt. } \angle = 3{\cdot}467 \times 90^\circ = 312{\cdot}03^\circ;$	3·467
$\cdot 03^\circ = \cdot 03 \times 60' = 1{\cdot}8';$	90
$\cdot 8' = \cdot 8 \times 60'' = 48''.$	312·03
$\therefore \text{ result} = 312^\circ\, 1'\, 48''.$	60
	1·8
	60
	48·

4. Angles of any size.

Suppose AOA' and BOB' to be two fixed lines meeting at right angles in O, and suppose a revolving line OP (turning about a fixed point at O) to start from OA and revolve in a direction ↺, *i.e.* opposite to that of the hands of a watch.



For any position of the revolving line between OA and OB, such as OP_1, it will have turned through an angle AOP_1, which is less than a right angle.

For any position between OB and OA', such as OP_2, the angle AOP_2 through which it has turned is greater than a right angle.

For any position OP_3, between OA' and OB', the angle traced out is AOP_3, *i.e.* $AOB + BOA' + A'OP_3$, *i.e.* two right angles $+ A'OP_3$, so that the angle described is greater than two right angles.

For any position OP_4, between OB' and OA, the angle turned through is similarly greater than three right angles.

When the revolving line has made a complete revolution, so that it coincides once more with OA, the angle through which it has turned is four right angles.

If the line OP still continue to revolve, the angle through which it has turned, when it is for the second time in the position OP_1, is not AOP_1 but four right angles $+ AOP_1$.

Similarly, when the revolving line, having made two complete revolutions, is once more in the position OP_2, the angle it has traced out is eight right angles $+ AOP_2$.

5. If the revolving line OP be between OA and OB, it is said to be in the first quadrant; if it be between OB and OA', it is in the second quadrant; if between OA' and OB', it is in the third quadrant; if it is between OB' and OA, it is in the fourth quadrant.

6. **Ex.** *What is the position of the revolving line when it has turned through* (1) 225°, (2) 480°, *and* (3) 1050°?

(1) Since $225° = 180° + 45°$, the revolving line has turned through 45° more than two right angles, and it is therefore in the third quadrant and halfway between OA' and OB'.

(2) Since $480° = 360° + 120°$, the revolving line has turned through 120° more than one complete revolution, and is therefore in the second quadrant, *i.e.* between OB and OA', and makes an angle of 30° with OB.

(3) Since $1050° = 11 \times 90° + 60°$, the revolving line has turned through 60° more than eleven right angles, and is therefore in the fourth quadrant, *i.e.* between OB' and OA, and makes 60° with OB'.

EXAMPLES. I.

Express in terms of a right angle the angles

1. 60°. 2. 75° 15′. 3. 63° 17′ 25″.

4. 130° 30′. 5. 210° 30′ 30″. 6. 370° 20′ 48″.

Mark the position of the revolving line when it has traced out the following angles:

7. $\frac{4}{3}$ right angle. 8. $3\frac{1}{2}$ right angles. 9. $13\frac{1}{3}$ right angles.

10. 120°. 11. 315°. 12. 745°. 13. 1185°.

14. How many degrees, minutes and seconds are respectively passed over in $11\frac{1}{9}$ minutes by the hour and minute hands of a watch?

15. One angle of a triangle is 80° 12′ 45″, and one of the other angles is equal to twice the third angle; find the other two angles.

16. The sum of the base angles of a triangle is 118° 27′ 23″ and the difference is 29° 43′ 29″; find all the angles of the triangle.

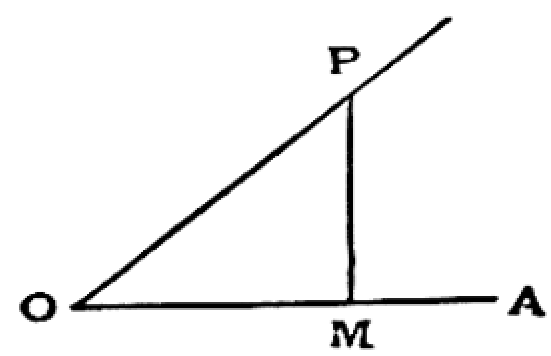
17. Three angles of a quadrilateral are 103° 15′ 25″, 73° 22′ 37″, and 45° 17′; find the fourth angle.

CHAPTER II.

TRIGONOMETRICAL RATIOS FOR ANGLES LESS THAN A RIGHT ANGLE.

7. In the present chapter we shall only consider angles which are less than a right angle.

Let a revolving line OP start from OA and revolve into the position OP, thus tracing out the angle AOP.



In the revolving line take any point P and draw PM perpendicular to the initial line OA.

In the triangle MOP, OP is the hypothenuse, PM is the perpendicular, and OM is the base.

The trigonometrical ratios, or functions, of the angle AOP are defined as follows:

$\dfrac{MP}{OP}$, *i.e.* $\dfrac{\text{Perp.}}{\text{Hyp.}}$, is called the **Sine** of the angle AOP;

$\dfrac{OM}{OP}$, *i.e.* $\dfrac{\text{Base}}{\text{Hyp.}}$, " " **Cosine** " "

$\dfrac{MP}{OM}$, *i.e.* $\dfrac{\text{Perp.}}{\text{Base}}$, " " **Tangent** " "

$\dfrac{OM}{MP}$, *i.e.* $\dfrac{\text{Base}}{\text{Perp.}}$, " " **Cotangent** " "

$\dfrac{OP}{MP}$, *i.e.* $\dfrac{\text{Hyp.}}{\text{Perp.}}$, " " **Cosecant** " "

$\dfrac{OP}{OM}$, *i.e.* $\dfrac{\text{Hyp.}}{\text{Base}}$, " " **Secant** " "

The quantity by which the cosine falls short of unity, *i.e.* $1-\cos AOP$, is called the **Versed Sine** of AOP; also the quantity $1-\sin AOP$, by which the sine falls short of unity, is called the **Coversed Sine** of AOP.

8. It will be noted that the trigonometrical ratios are all **numbers.**

The names of these eight ratios are written, for brevity, $\sin AOP$, $\cos AOP$, $\tan AOP$, $\cot AOP$, $\operatorname{cosec} AOP$, $\sec AOP$, $\operatorname{vers} AOP$, and $\operatorname{covers} AOP$ respectively.

The two latter ratios are seldom used.

9. It will be noticed, from the definitions, that the cosecant is the reciprocal of the sine, so that

$$\operatorname{cosec} AOP=\frac{1}{\sin AOP}.$$

So the secant is the reciprocal of the cosine, *i.e.*

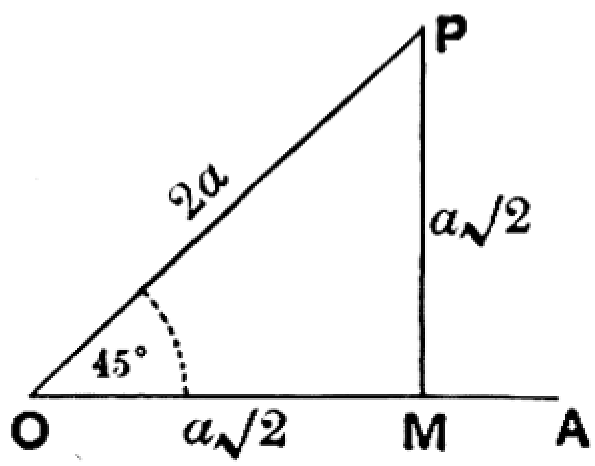
$$\sec AOP=\frac{1}{\cos AOP},$$

and the cotangent is the reciprocal of the tangent, *i.e.*

$$\cot AOP=\frac{1}{\tan AOP}.$$

Values of the trigonometrical ratios in some useful cases.

10. *Angle of* 45°.



Let the angle AOP traced out be 45°.

Then, since the three angles of a triangle are together equal to two right angles,

$$\angle OPM=180°-\angle POM-\angle PMO$$
$$=180°-45°-90°=45°=\angle POM.$$

$$\therefore\ OM=MP.$$

If OP be called $2a$, we then have

$$4a^2 = OP^2 = OM^2 + MP^2 = 2 \, . \, OM^2,$$

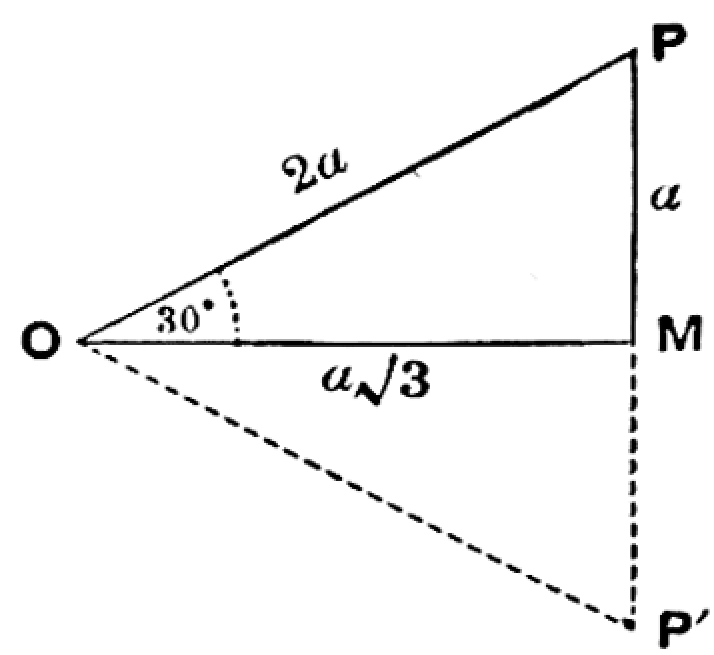
so that
$$OM = a\sqrt{2}.$$

$$\therefore \sin 45^\circ = \frac{MP}{OP} = \frac{a\sqrt{2}}{2a} = \frac{1}{\sqrt{2}},$$

$$\cos 45^\circ = \frac{OM}{OP} = \frac{a\sqrt{2}}{2a} = \frac{1}{\sqrt{2}},$$

and
$$\tan 45^\circ = \frac{MP}{OM} = \frac{a\sqrt{2}}{a\sqrt{2}} = 1.$$

11. *Angle of* 30°.



Let the angle MOP traced out be 30°.

Produce PM to P' making MP' equal to PM.

The two triangles OMP and OMP' have their sides OM and MP' equal to OM and MP, and also the contained angles equal.

Therefore $OP' = OP$, and $\angle OP'P = \angle OPP' = 60°$, so that the triangle $P'OP$ is equilateral.

Hence, if OP be called $2a$, we have

$$MP = \tfrac{1}{2}P'P = \tfrac{1}{2}OP = a.$$

Also
$$OM = \sqrt{OP^2 - MP^2} = \sqrt{4a^2 - a^2} = a\sqrt{3}.$$

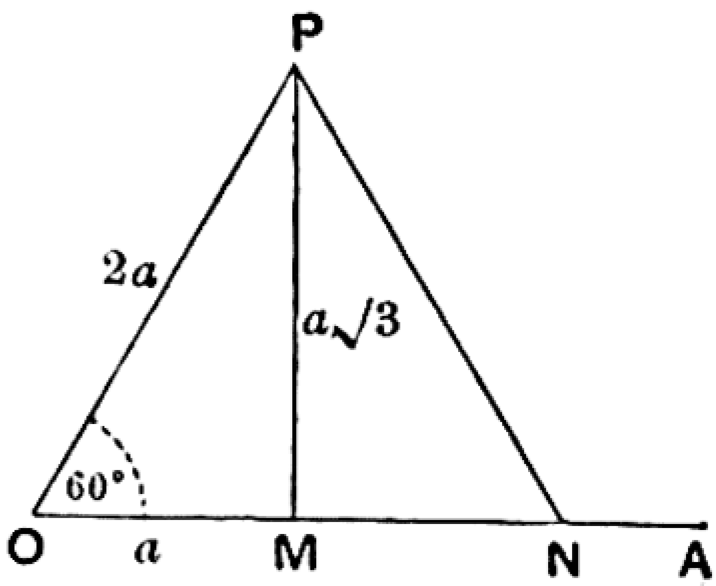
$$\therefore \sin 30^\circ = \frac{MP}{OP} = \frac{1}{2},$$

$$\cos 30^\circ = \frac{OM}{OP} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2},$$

and
$$\tan 30^\circ = \frac{MP}{OM} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

12. *Angle of* 60°.

Let the angle AOP traced out be 60°.



Take a point N on OA, so that

$$MN = OM = a \text{ (say)}.$$

The two triangles OMP and NMP have now the sides OM and MP equal to NM and MP respectively, and the included angles equal, so that the triangles are equal.

$$\therefore\ PN = OP, \text{ and } \angle PNM = \angle POM = 60°.$$

The triangle OPN is therefore equilateral, and hence

$$OP = ON = 2OM = 2a.$$

$$\therefore\ MP = \sqrt{OP^2 - OM^2} = \sqrt{4a^2 - a^2} = a\sqrt{3}.$$

Hence $$\sin 60° = \frac{MP}{OP} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2},$$

$$\cos 60° = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2},$$

and $$\tan 60° = \frac{MP}{OM} = \frac{a\sqrt{3}}{a} = \sqrt{3}.$$

13. *Angle of* 0°.

Let the revolving line OP have turned through a very small angle, so that the angle MOP is very small.

The magnitude of MP is then very small, and initially, before OP had turned through an angle large enough to be perceived, the quantity MP was smaller than any quantity we could assign, *i.e.* was what we denote by 0.

Also, in this case, the two points M and P very nearly coincide, and the smaller the angle AOP the more nearly do they coincide.

Hence, when the angle AOP is actually zero, the two lengths OM and OP are equal and MP is zero.

Hence
$$\sin 0^\circ = \frac{MP}{OP} = \frac{0}{OP} = 0,$$
$$\cos 0^\circ = \frac{OM}{OP} = \frac{OP}{OP} = 1,$$
and
$$\tan 0^\circ = \frac{MP}{OM} = \frac{0}{OP} = 0.$$

Also $\cot 0^\circ =$ the value of $\frac{OM}{MP}$ when M and P coincide

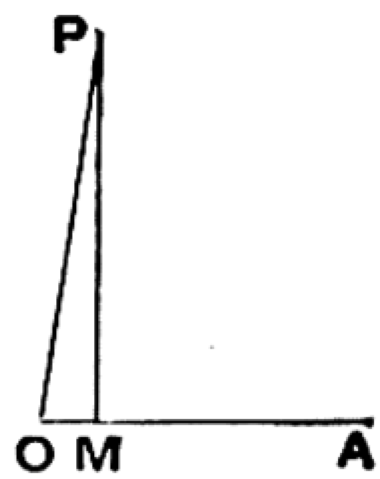
= the ratio of a finite quantity to something infinitely small

= a quantity which is infinitely great.

Such a quantity is usually denoted by the symbol ∞.

Hence
$$\cot 0^\circ = \infty.$$
Similarly
$$\operatorname{cosec} 0^\circ = \frac{OP}{MP} = \infty \text{ also.}$$
And
$$\sec 0^\circ = \frac{OP}{OM} = 1.$$

14. *Angle of* 90°.



Let the angle AOP be very nearly, but not quite, a right angle.

When OP has actually described a right angle, the point M coincides with O, so that then OM is zero and OP and MP are equal.

Hence
$$\sin 90^\circ = \frac{MP}{OP} = \frac{OP}{OP} = 1,$$
$$\cos 90^\circ = \frac{OM}{OP} = \frac{0}{OP} = 0,$$
$$\tan 90^\circ = \frac{MP}{OM} = \frac{\text{a finite quantity}}{\text{an infinitely small quantity}}$$
$$= \text{a number infinitely large} = \infty,$$

$$\cot 90^\circ = \frac{OM}{MP} = \frac{0}{MP} = 0,$$

$$\sec 90^\circ = \frac{OP}{OM} = \infty \text{, as in the case of the tangent,}$$

and
$$\operatorname{cosec} 90^\circ = \frac{OP}{MP} = \frac{OP}{OP} = 1.$$

EXAMPLES. II.

Verify that

1. $\sin^2 30^\circ + \sin^2 45^\circ + \sin^2 60^\circ = \frac{3}{2}$.

2. $\tan^2 30^\circ + \tan^2 45^\circ + \tan^2 60^\circ = 4\frac{1}{3}$.

3. $\cos^2 0^\circ + \cos^2 45^\circ + \cos^2 90^\circ = \frac{3}{2}$.

4. $\sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ = 1$.

5. $\cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ = -\frac{\sqrt{3}-1}{2\sqrt{2}}$.

6. $\sin^2 60^\circ - \sin^2 30^\circ = \frac{1}{6} \tan 60^\circ \cot 30^\circ$.

7. $\frac{1}{3} \cot^2 30^\circ + 3 \sin^2 60^\circ - 2 \operatorname{cosec}^2 60^\circ - \frac{3}{4} \tan^2 30^\circ = 3\frac{1}{3}$.

8. $\operatorname{cosec}^2 45^\circ \,.\, \sec^2 30^\circ \,.\, \sin^3 90^\circ \,.\, \cos 60^\circ = 1\frac{1}{3}$.

9. $4 \cot^2 45^\circ - \sec^2 60^\circ + \sin^3 30^\circ = \frac{1}{8}$.

10. $\dfrac{\cos 60^\circ + \cos 30^\circ}{\sec 60^\circ + \operatorname{cosec} 60^\circ} = \dfrac{\sqrt{3}}{4}$.

11. If $A = 30^\circ$, verify that

    (1) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$,

    (2) $\sin 2A = 2 \sin A \cos A$,

    (3) $\cos 3A = 4 \cos^3 A - 3 \cos A$,

    (4) $\sin 3A = 3 \sin A - 4 \sin^3 A$,

    and (5) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$.

12. If $A = 45^\circ$, verify that

    (1) $\sin 2A = 2 \sin A \cos A$,

    (2) $\cos 2A = 1 - 2 \sin^2 A$,

    and (3) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$.

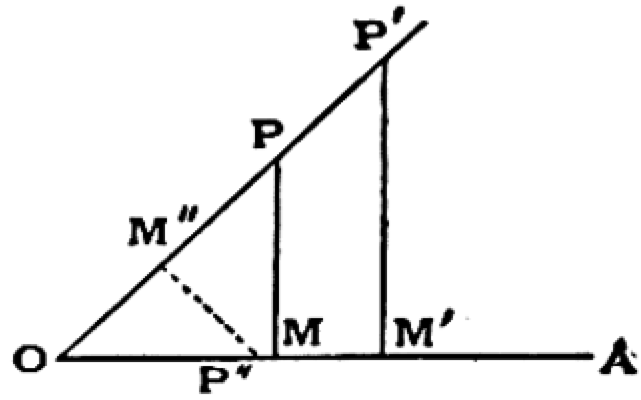
15. For every value of the angle AOP the trigonometrical relations, as defined in Art. 7, have definite numerical values. These values are tabulated for consecutive values of the angles. Thus Table II. at the end of this book contains the values (correct to four places of decimals) of the sines of angles less than a right angle. Thus its eleventh line tells us that $\sin 10° = \cdot 1736$.

So Table III. gives us the tangents of the same series of angles. For example, $\tan 15° = \cdot 2679$.

By means of the relations proved in the next few articles these two tables give us the other trigonometrical ratios.

16. *To shew that the trigonometrical ratios are always the same for the same angle.*

We have to shew that, if in the revolving line OP any other point P' be taken and $P'M'$ be drawn perpendicular to OA, the ratios derived from the triangle $OP'M'$ are the same as those derived from the triangle OPM.



In the two triangles, the angle at O is common, and the angles at M and M' are both right angles and therefore equal.

Hence the two triangles are equiangular and therefore, by geometry, we have

$$\frac{MP}{OP} = \frac{M'P'}{OP'},$$

i.e. the sine of the angle AOP is the same whatever point we take on the revolving line.

Since, by the same proposition, we have

$$\frac{OM}{OP} = \frac{OM'}{OP'} \text{ and } \frac{MP}{OM} = \frac{M'P'}{OM'},$$

it follows that the cosine and tangent are the same whatever point be taken on the revolving line. Similarly for the other ratios.

If OA be considered as the revolving line, and in it be taken any point P'' and $P''M''$ be drawn perpendicular to OP, the functions as derived from the triangle $OP''M''$ will have the same values as before.

For, since in the two triangles OPM and $OP''M''$, the two angles $P''OM''$ and $OM''P''$ are respectively equal to POM and OMP, these two triangles are equiangular and therefore similar, and we have

$$\frac{M''P''}{OP''} = \frac{MP}{OP}, \quad \text{and} \quad \frac{OM''}{OP''} = \frac{OM}{OP}.$$

17. *Fundamental relations between the trigonometrical ratios of an angle.*

We shall find that if one of the trigonometrical ratios of an angle be known, the numerical magnitude of each of the others is known also.

Let the angle AOP (Fig., Art. 7) be denoted by θ, [pronounced "Theta"].

In the right-angled triangle MOP we have

$$MP^2 + OM^2 = OP^2 \text{..................} (1).$$

Hence, dividing by OP^2, we have

$$\left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = 1,$$

i.e. $$(\sin\theta)^2 + (\cos\theta)^2 = 1.$$

The quantity $(\sin\theta)^2$ is always written $\sin^2\theta$, and so for the other ratios.

Hence this relation is

$$\boldsymbol{\sin^2\theta + \cos^2\theta = 1} \text{..................} (2).$$

Again, dividing both sides of equation (1) by OM^2, we have

$$\left(\frac{MP}{OM}\right)^2 + 1 = \left(\frac{OP}{OM}\right)^2,$$

i.e. $$(\tan\theta)^2 + 1 = (\sec\theta)^2,$$

so that $$\boldsymbol{\sec^2\theta = 1 + \tan^2\theta} \text{..................} (3).$$

Again, dividing equation (1) by MP^2, we have

$$1+\left(\frac{OM}{MP}\right)^2=\left(\frac{OP}{MP}\right)^2,$$

i.e. $$1+(\cot\theta)^2=(\operatorname{cosec}\theta)^2,$$

so that $$\mathbf{cosec^2\theta=1+cot^2\theta} \qquad\text{..............(4).}$$

Also, since $\sin\theta=\dfrac{MP}{OP}$ and $\cos\theta=\dfrac{OM}{OP}$,

we have $$\frac{\sin\theta}{\cos\theta}=\frac{MP}{OP}\div\frac{OM}{OP}=\frac{MP}{OM}=\tan\theta.$$

Hence $$\mathbf{\tan\theta=\frac{\sin\theta}{\cos\theta}} \qquad\text{..................(5).}$$

Similarly $$\mathbf{\cot\theta=\frac{\cos\theta}{\sin\theta}} \qquad\text{..................(6).}$$

18. *Limits to the values of the trigonometrical ratios.*

From equation (2) of Art. 17, we have

$$\sin^2\theta+\cos^2\theta=1.$$

Now $\sin^2\theta$ and $\cos^2\theta$, being both squares, are both necessarily positive. Hence, since their sum is unity, neither of them can be greater than unity.

[For if one of them, say $\sin^2\theta$, were greater than unity, the other, $\cos^2\theta$, would have to be negative, which is impossible.]

Hence neither the sine nor the cosine can be numerically greater than unity.

Since $\sin\theta$ cannot be greater than unity, therefore $\operatorname{cosec}\theta$, which equals $\dfrac{1}{\sin\theta}$, cannot be numerically less than unity.

So $\sec\theta$, which equals $\dfrac{1}{\cos\theta}$, cannot be numerically less than unity.

19. The results of the previous article follow easily from the figure of Art. 7.

For, whatever be the value of the angle AOP, neither the side OM nor the side MP is ever greater than OP.

Since MP is never greater than OP, the ratio $\frac{MP}{OP}$ is never greater than unity, so that the sine of an angle is never greater than unity.

Also, since OM is never greater than OP, the ratio $\frac{OM}{OP}$ is never greater than unity, *i.e.* the cosine is never greater than unity.

20. The fundamental relations of the preceding article enable very considerable simplifications to be made in trigonometrical formulae. It is important that the student should acquire considerable facility in their use.

Ex. 1. *Prove that* $\sqrt{\frac{1-\cos A}{1+\cos A}} = \operatorname{cosec} A - \cot A$.

We have
$$\sqrt{\frac{1-\cos A}{1+\cos A}} = \sqrt{\frac{(1-\cos A)^2}{1-\cos^2 A}}$$
$$= \frac{1-\cos A}{\sqrt{1-\cos^2 A}} = \frac{1-\cos A}{\sin A},$$
by relation (2) of the last article,
$$= \frac{1}{\sin A} - \frac{\cos A}{\sin A} = \operatorname{cosec} A - \cot A.$$

Ex. 2. *Prove that*
$$\sqrt{\sec^2 A + \operatorname{cosec}^2 A} = \tan A + \cot A.$$

We have seen that $\sec^2 A = 1 + \tan^2 A$,

and $\operatorname{cosec}^2 A = 1 + \cot^2 A$.
$$\therefore \sec^2 A + \operatorname{cosec}^2 A = \tan^2 A + 2 + \cot^2 A$$
$$= \tan^2 A + 2\tan A \cot A + \cot^2 A$$
$$= (\tan A + \cot A)^2,$$
so that
$$\sqrt{\sec^2 A + \operatorname{cosec}^2 A} = \tan A + \cot A.$$

Ex. 3. *Prove that*
$$(\operatorname{cosec} A - \sin A)(\sec A - \cos A)(\tan A + \cot A) = 1.$$

The given expression

$$= \left(\frac{1}{\sin A} - \sin A\right)\left(\frac{1}{\cos A} - \cos A\right)\left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)$$

$$= \frac{1 - \sin^2 A}{\sin A} \cdot \frac{1 - \cos^2 A}{\cos A} \cdot \frac{\sin^2 A + \cos^2 A}{\sin A \cos A}$$

$$= \frac{\cos^2 A}{\sin A} \cdot \frac{\sin^2 A}{\cos A} \cdot \frac{1}{\sin A \cos A}$$

$$= 1.$$

EXAMPLES. III.

Prove the following statements for any angle A.

1. $(\cos A + \sin A)^2 + (\cos A - \sin A)^2 = 2.$

2. $\cos^4 A - \sin^4 A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$

3. $\sin^3 A + \cos^3 A = (\sin A + \cos A)(1 - \sin A \cos A).$

4. $\sin^4 A + \cos^4 A = 1 - 2\sin^2 A \cos^2 A.$

5. $\dfrac{\sin A}{1 + \cos A} + \dfrac{1 + \cos A}{\sin A} = 2 \operatorname{cosec} A.$

6. $\sqrt{\dfrac{1 - \sin A}{1 + \sin A}} = \sec A - \tan A.$

7. $\dfrac{\operatorname{cosec} A}{\operatorname{cosec} A - 1} + \dfrac{\operatorname{cosec} A}{\operatorname{cosec} A + 1} = 2\sec^2 A.$

8. $\tan A + \cot A = \sec A \operatorname{cosec} A.$

9. $\dfrac{1}{\sec A - \tan A} = \sec A + \tan A.$

10. $\dfrac{1 - \tan A}{1 + \tan A} = \dfrac{\cot A - 1}{\cot A + 1}.$

11. $\dfrac{1}{1 + \tan^2 A} + \dfrac{1}{1 + \cot^2 A} = 1.$

12. $(\cot A - 1)^2 + (\cot A + 1)^2 = 2\operatorname{cosec}^2 A.$

13. $\dfrac{1 + \tan^2 A}{1 + \cot^2 A} = \dfrac{\sin^2 A}{\cos^2 A}.$

14. $(\sec A + \cos A)(\sec A - \cos A) = \tan^2 A + \sin^2 A.$

15. $\sec^4 A - \tan^4 A = \sec^2 A + \tan^2 A.$

16. $\operatorname{cosec}^4 A - \cot^4 A = \operatorname{cosec}^2 A + \cot^2 A.$

17. $\dfrac{\operatorname{cosec} A}{\cot A + \tan A} = \cos A.$

18. $\dfrac{\cos A}{1-\tan A}+\dfrac{\sin A}{1-\cot A}=\sin A+\cos A.$

19. $(\sin A+\cos A)(\cot A+\tan A)=\sec A+\operatorname{cosec} A.$

20. $\sqrt{\operatorname{cosec}^2 A-1}=\cos A\operatorname{cosec} A.$

21. $\dfrac{\cot A-1}{\cot A+1}+\dfrac{\cot A+1}{\cot A-1}=\dfrac{2}{\cos^2 A-\sin^2 A}.$

22. $\tan^2 A+\cot^2 A+2=\sec^2 A\operatorname{cosec}^2 A.$

23. $\tan^2 A-\sin^2 A=\sin^4 A\sec^2 A.$

24. $\cos^6 A+\sin^6 A=1-3\sin^2 A\cos^2 A.$

25. $\sin^8 A-\cos^8 A=(\sin^2 A-\cos^2 A)(1-2\sin^2 A\cos^2 A).$

26. $(\sin\alpha+\operatorname{cosec}\alpha)^2+(\cos\alpha+\sec\alpha)^2=\tan^2\alpha+\cot^2\alpha+7.$

27. $2\operatorname{versin} A+\cos^2 A=1+\operatorname{versin}^2 A.$

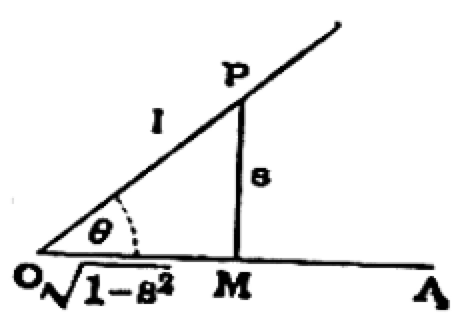
21. We can express the trigonometrical ratios of an angle in terms of any one of them.

The simplest method of procedure is best shewn by examples.

Ex. 1. *To express all the trigonometrical ratios in terms of the sine.*

Let AOP be any angle θ.

Let the length OP be unity and let the corresponding length of MP be s.



Then $\qquad OM=\sqrt{OP^2-MP^2}=\sqrt{1-s^2}.$

Hence $\qquad \sin\theta=\dfrac{MP}{OP}=\dfrac{s}{1}=s,$

$$\cos\theta=\frac{OM}{OP}=\sqrt{1-s^2}=\sqrt{1-\sin^2\theta},$$

$$\tan\theta=\frac{MP}{OM}=\frac{s}{\sqrt{1-s^2}}=\frac{\sin\theta}{\sqrt{1-\sin^2\theta}},$$

$$\cot\theta=\frac{OM}{MP}=\frac{\sqrt{1-s^2}}{s}=\frac{\sqrt{1-\sin^2\theta}}{\sin\theta},$$

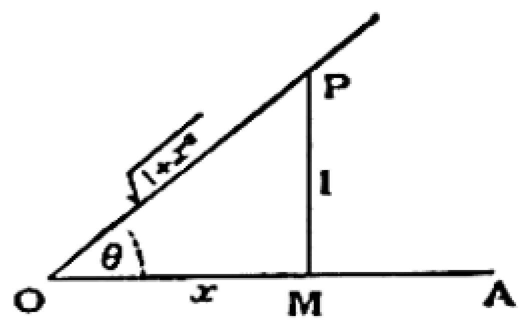
$$\operatorname{cosec}\theta=\frac{OP}{MP}=\frac{1}{s}=\frac{1}{\sin\theta},$$

and $$\sec\theta=\frac{OP}{OM}=\frac{1}{\sqrt{1-s^2}}=\frac{1}{\sqrt{1-\sin^2\theta}}.$$

The last five equations give what is required.

Ex. 2. *To express all the trigonometrical ratios in terms of the cotangent.*

Taking the usual figure, let the length MP be unity, and let the corresponding value of OM be x.



By geometry,

$$OP=\sqrt{OM^2+MP^2}=\sqrt{1+x^2}.$$

Hence
$$\cot\theta=\frac{OM}{MP}=\frac{x}{1}=x,$$

$$\sin\theta=\frac{MP}{OP}=\frac{1}{\sqrt{1+x^2}}=\frac{1}{\sqrt{1+\cot^2\theta}},$$

$$\cos\theta=\frac{OM}{OP}=\frac{x}{\sqrt{1+x^2}}=\frac{\cot\theta}{\sqrt{1+\cot^2\theta}},$$

$$\tan\theta=\frac{MP}{OM}=\frac{1}{x}=\frac{1}{\cot\theta},$$

$$\sec\theta=\frac{OP}{OM}=\frac{\sqrt{1+x^2}}{x}=\frac{\sqrt{1+\cot^2\theta}}{\cot\theta},$$

and
$$\operatorname{cosec}\theta=\frac{OP}{MP}=\frac{\sqrt{1+x^2}}{1}=\sqrt{1+\cot^2\theta}.$$

The last five equations give what is required.

It will be noticed that, in each case, the denominator of the fraction which defines the trigonometrical ratio was taken equal to unity. For example, the sine is $\frac{MP}{OP}$, and hence in Ex. 1 the denominator OP is taken equal to unity.

The cotangent is $\frac{OM}{MP}$, and hence in Ex. 2 the side MP is taken equal to unity.

Similarly suppose we had to express the other ratios in terms of the cosine, we should, since the cosine is equal to $\frac{OM}{OP}$, put OP equal to unity and OM equal to c. The working would then be similar to that of Exs. 1 and 2.

In the following examples the sides have numerical values.

Ex. 3. *If* $\cos\theta$ *equal* $\frac{3}{5}$, *find the values of the other ratios.*

Along the initial line OA take OM equal to 3, and erect a perpendicular MP.

Let a line OP, of length 5, revolve round O until its other end meets this perpendicular in the point P. Then AOP is the angle θ.

By geometry, $MP=\sqrt{OP^2-OM^2}=\sqrt{5^2-3^2}=4.$

Hence clearly

$$\sin\theta=\frac{4}{5},\quad \tan\theta=\frac{4}{3},\quad \cot\theta=\frac{3}{4},\quad \operatorname{cosec}\theta=\frac{5}{4},\quad \text{and}\quad \sec\theta=\frac{5}{3}.$$

Ex. 4. *Supposing θ to be an angle whose sine is $\frac{1}{3}$, to find the numerical magnitude of the other trigonometrical ratios.*

Here $\sin\theta=\frac{1}{3}$, so that the relation (2) of Art. 17 gives

$$\left(\frac{1}{3}\right)^2+\cos^2\theta=1,$$

i.e. $$\cos^2\theta=1-\frac{1}{9}=\frac{8}{9},$$

i.e. $$\cos\theta=\frac{2\sqrt{2}}{3}=\frac{2}{3}\times 1{\cdot}4142\ldots=\cdot 9428\ldots$$

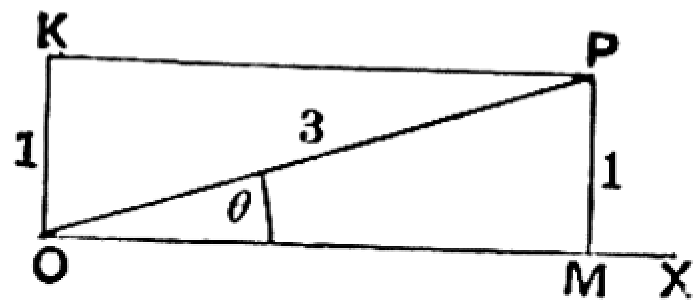
Hence $$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}=\cdot 3535\ldots,$$

$$\cot\theta=\frac{1}{\tan\theta}=2\sqrt{2}=2{\cdot}8284\ldots,$$

$$\operatorname{cosec}\theta=\frac{1}{\sin\theta}=3,$$

$$\sec\theta=\frac{1}{\cos\theta}=\frac{3}{2\sqrt{2}}=\frac{3\sqrt{2}}{4}=1{\cdot}0606\ldots.$$

By drawing. Take OX as initial line and erect a perpendicular OK to it, one inch in length. Through K draw KP perpendicular to OK. With centre O, and radius 3 inches, describe a circle to meet KP in P. Draw PM perpendicular to OX. Then XOP is the angle θ required. On measuring OM, we find it to be 2·83 inches.



$$\therefore \cos\theta=\frac{OM}{OP}=\frac{2{\cdot}83}{3}=\cdot 94\ldots,$$

and so for the other ratios as above.

22. In the following table is given the result of expressing each trigonometrical ratio in terms of each of the others.

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\dfrac{1}{\sqrt{1+\cot^2\theta}}$	$\dfrac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\dfrac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\dfrac{1}{\sqrt{1+\tan^2\theta}}$	$\dfrac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\dfrac{1}{\sec\theta}$	$\dfrac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
$\tan\theta$	$\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\dfrac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\dfrac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\cot\theta$	$\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\dfrac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\dfrac{1}{\tan\theta}$	$\cot\theta$	$\dfrac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\operatorname{cosec}^2\theta-1}$
$\sec\theta$	$\dfrac{1}{\sqrt{1-\sin^2\theta}}$	$\dfrac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\dfrac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\dfrac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\operatorname{cosec}\theta$	$\dfrac{1}{\sin\theta}$	$\dfrac{1}{\sqrt{1-\cos^2\theta}}$	$\dfrac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\dfrac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\operatorname{cosec}\theta$

EXAMPLES. IV.

1. Express all the other trigonometrical ratios in terms of the cosine.

2. Express all the ratios in terms of the tangent.

3. Express all the ratios in terms of the cosecant.

4. Express all the ratios in terms of the secant.

5. The sine of a certain angle is $\frac{1}{4}$; find the numerical values of the other trigonometrical ratios of this angle.

6. If $\sin\theta = \frac{12}{13}$, find $\tan\theta$. Verify by an accurate figure.

7. If $\sin A = \frac{11}{61}$, find $\tan A$, $\cos A$, and $\sec A$.

8. If $\cos\theta = \frac{4}{5}$, find $\sin\theta$ and $\cot\theta$. Verify by an accurate graph.

9. If $\cos A = \frac{9}{41}$, find $\tan A$ and $\operatorname{cosec} A$.

10. If $\tan\theta = \frac{3}{4}$, find the sine, cosine and cosecant of θ.

11. Construct to scale an angle whose sine is $\frac{3}{5}$ and another angle whose cosine is $\cdot 35$. Find the tangent of each angle to three places of decimals.

12. Find to two places of decimals the cosecant of the angle whose cotangent is $2\cdot 3$, and the secant of the angle whose tangent is $3\cdot 5$. Verify by a graph and measurement.

13. If $\sec A = \frac{3}{2}$, find $\tan A$ and $\operatorname{cosec} A$.

14. If $\sin\theta + \operatorname{cosec}\theta = 2\frac{1}{2}$, find the values of $\sin\theta$.

15. If $8\sin\theta = 4 + \cos\theta$, find $\sin\theta$.

16. If $\tan\theta + \sec\theta = 1\cdot 5$, find $\sin\theta$.

17. If $\cot\theta + \operatorname{cosec}\theta = 5$, find $\cos\theta$.

18. If $3\sec^4\theta + 8 = 10\sec^2\theta$, find the values of $\tan\theta$.

19. If $\tan^2\theta + \sec\theta = 5$, find $\cos\theta$.

20. If $3\tan^2\theta + \cot^2\theta = 4$, find the values of $\tan\theta$.

21. If $\sec^2\theta = 2 + 2\tan\theta$, find $\tan\theta$.

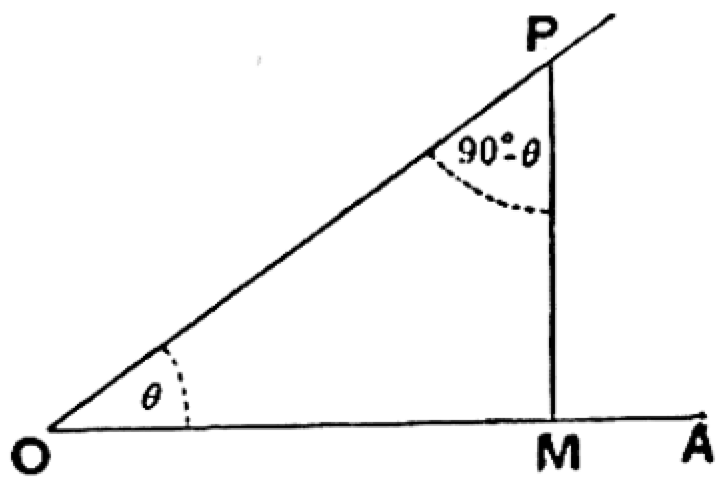
22. If $\tan\theta = \frac{2x(x+1)}{2x+1}$, find $\sin\theta$ and $\cos\theta$.

23. Complementary Angles. Def. Two angles are said to be complementary when their sum is equal to a right angle. Thus, any angle θ and the angle $90^\circ - \theta$ are complementary.

The complement of 30° is $90^\circ - 30^\circ$, *i.e.* 60°. The complement of $20^\circ\,45'$ is $90^\circ - 20^\circ\,45'$, *i.e.* $69^\circ\,15'$.

24. *To find the relations between the trigonometrical ratios of two complementary angles.*

Let the revolving line, starting from OA, trace out any acute angle AOP, equal to θ. From any point P on it draw PM perpendicular to OA.



Since the three angles of a triangle are together equal to two right angles, and since OMP is a right angle, the sum of the two angles MOP and OPM is a right angle.

They are therefore complementary and

$$\angle OPM = 90^\circ - \theta.$$

[When the angle OPM is considered, the line PM is the "base" and MO is the "perpendicular."]

We then have

$$\mathbf{sin}\,(90^\circ - \boldsymbol{\theta}) = \sin MPO = \frac{MO}{PO} = \cos AOP = \mathbf{cos}\,\boldsymbol{\theta},$$

$$\mathbf{cos}\,(90^\circ - \boldsymbol{\theta}) = \cos MPO = \frac{PM}{PO} = \sin AOP = \mathbf{sin}\,\boldsymbol{\theta},$$

$$\tan(90^\circ - \theta) = \tan MPO = \frac{MO}{PM} = \cot AOP = \cot\theta,$$

$$\cot(90^\circ - \theta) = \cot MPO = \frac{PM}{MO} = \tan AOP = \tan\theta,$$

$$\operatorname{cosec}(90^\circ - \theta) = \operatorname{cosec} MPO = \frac{PO}{MO} = \sec AOP = \sec\theta,$$

and

$$\sec(90^\circ - \theta) = \sec MPO = \frac{PO}{PM} = \operatorname{cosec} AOP = \operatorname{cosec}\theta.$$

Hence we observe that

the Sine of any angle = the Cosine of its complement,
the Tangent of any angle = the Cotangent of its complement,
and the Secant of an angle = the Cosecant of its complement.

From this is apparent what is the derivation of the names **Co**sine, **Co**tangent, and **Co**secant.

25. The student is advised before proceeding any further to make himself quite familiar with the following Table. [For an extension of this table, see Art. 43.]

Angle	0^0	30^0	45^0	60^0	90^0
Sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
Cotangent	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
Cosecant	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
Secant	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

If the student commits accurately to memory the portion of the above table included between thick lines, he should be able to easily reproduce the rest.

For

(1) the sines of 60° and 90° are respectively the cosines of 30° and 0°.
(Art. 24.)

(2) the cosines of 60° and 90° are respectively the sines of 30° and 0°. (Art. 24.)

Hence the second and third lines are known.

(3) The tangent of any angle is the result of dividing the sine by the cosine.

Hence any quantity in the fourth line is obtained by dividing the corresponding quantity in the second line by the corresponding quantity in the third line.

(4) The cotangent of any angle is the reciprocal of the tangent, so that the quantities in the fifth row are the reciprocals of the corresponding quantities in the fourth row.

(5) Since $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, the sixth row is obtained by inverting the corresponding quantities in the second row.

(6) Since $\sec\theta = \frac{1}{\cos\theta}$, the seventh row is similarly obtained from the third row.

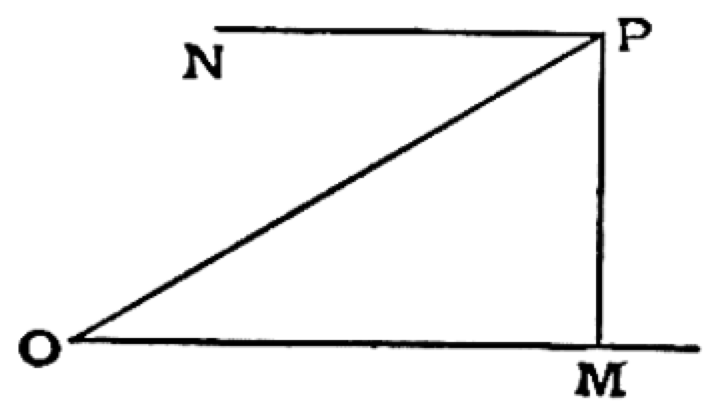
CHAPTER III.

SIMPLE PROBLEMS IN HEIGHTS AND DISTANCES.

26. ONE of the objects of Trigonometry is to find the distances between points, or the heights of objects, without actually measuring these distances or these heights.

27. Suppose O and P to be two points, P being at a higher level than O.

Let OM be a horizontal line drawn through O to meet in M the vertical line drawn through P.



The angle MOP is called the **Angle of Elevation** of the point P as seen from O.

Draw PN parallel to MO, so that PN is the horizontal line passing through P. The angle NPO is the **Angle of Depression** of the point O as seen from P.

28. Two of the instruments used in practical work are the Theodolite and the Sextant.

The Theodolite is used to measure angles in a vertical plane.

In its simple form, it consists of a telescope attached to a flat piece of wood. This piece of wood is supported by three legs and can be arranged so as to be accurately horizontal.

This table being at O and horizontal, and the telescope being initially pointing in the direction OM, the latter can be made to rotate in a vertical plane until it points accurately towards P. A graduated scale shews the angle through which it has been turned from the horizontal, *i.e.* gives us the angle of elevation MOP.

Similarly, if the instrument were at P, the angle NPO through which the telescope would have to be turned, downward from the horizontal, would give us the angle NPO.

The instrument can also be used to measure angles in a horizontal plane.

29. The Sextant is used to find the angle subtended by any two points D and E at a third point F. It is an instrument much used on board ships.

Its construction and application are too complicated to be here considered.

30. We shall now solve a few simple examples in heights and distances.

Ex. 1. *A vertical flagstaff stands on a horizontal plane; from a point distant* 150 *feet from its foot, the angle of elevation of its top is found to be* 30°; *find the height of the flagstaff.*

Let MP (Fig. Art. 27) represent the flagstaff and O the point from which the angle of elevation is taken.

Then $OM = 150$ feet, and $\angle MOP = 30°$.

Since PMO is a right angle, we have

$$\frac{MP}{OM} = \tan MOP = \tan 30° = \frac{1}{\sqrt{3}} \text{ (Art. 11)}.$$

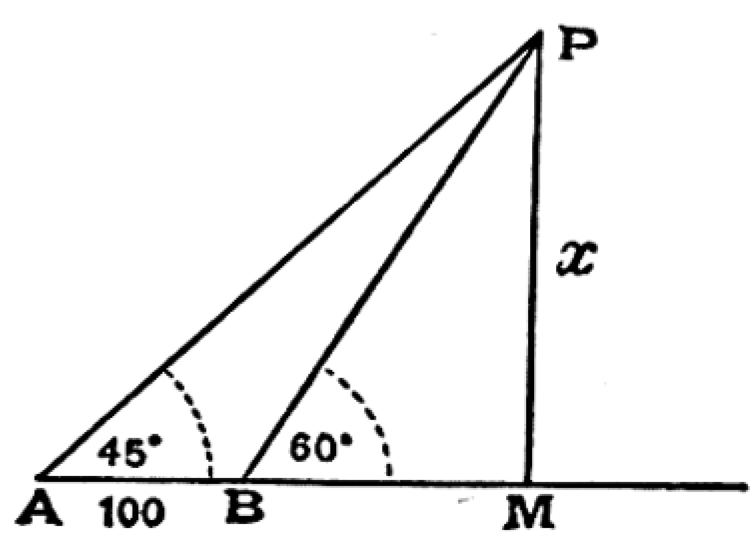
$$\therefore MP = \frac{OM}{\sqrt{3}} = \frac{150}{\sqrt{3}} = \frac{150\sqrt{3}}{3} = 50\sqrt{3}.$$

Now, by extraction of the square root, we have

$$\sqrt{3} = 1{\cdot}73205\ldots$$

Hence $\qquad MP = 50 \times 1{\cdot}73205\ldots \text{ feet} = 86{\cdot}60\ldots\ldots \text{ feet}.$

Ex. 2. *A man wishes to find the height of a church spire which stands on a horizontal plane; at a point on this plane he finds the angle of elevation of the top of the spire to be* 45°; *on walking* 100 *feet toward the tower he finds the corresponding angle of elevation to be* 60°; *deduce the height of the tower and also the original distance of the man from the foot of the spire.*



Let P be the top of the spire and A and B the two points at which the angles of elevation are taken. Draw PM perpendicular to AB produced and let MP be x.

We are given $AB=100$ feet,

$$\angle MAP=45^\circ,$$

and

$$\angle MBP=60^\circ.$$

We then have

$$\frac{AM}{x}=\cot 45^\circ,$$

and

$$\frac{BM}{x}=\cot 60^\circ=\frac{1}{\sqrt{3}}.$$

Hence $$AM=x, \text{ and } BM=\frac{x}{\sqrt{3}}.$$

$$\therefore\ 100=AM-BM=x-\frac{x}{\sqrt{3}}=x\frac{\sqrt{3}-1}{\sqrt{3}}.$$

$$\therefore\ x=\frac{100\sqrt{3}}{\sqrt{3}-1}=\frac{100\sqrt{3}(\sqrt{3}+1)}{3-1}=50(3+\sqrt{3})$$

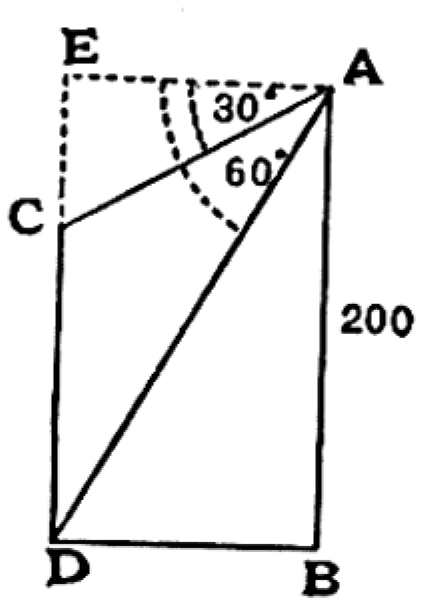
$$=50[3+1{\cdot}73205\ldots]=236{\cdot}6\ldots \text{ feet.}$$

Also $AM=x$, so that both of the required distances are equal to $236{\cdot}6\ldots$ feet.

Ex. 3. *From the top of a cliff, 200 feet high, the angles of depression of the top and bottom of a tower are observed to be* 30° *and* 60°*; find the height of the tower.*

Let A be the point of observation, BA the height of the cliff, and let CD be the tower.

Draw AE horizontally, so that $\angle EAC=30^\circ$ and $\angle EAD=60^\circ$.



Let x feet be the height of the tower; produce DC to meet AE in E, so that $CE=AB-x=200-x$.

Since $\angle ADB = \angle DAE = 60^\circ$ (Euc. I. 29),

$$\therefore\ DB = AB \cot ADB = 200 \cot 60^\circ = \frac{200}{\sqrt{3}}.$$

Also
$$\frac{200-x}{DB} = \frac{CE}{EA} = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

$$\therefore\ 200 - x = \frac{DB}{\sqrt{3}} = \frac{200}{3},$$

so that
$$x = 200 - \frac{200}{3} = 133\tfrac{1}{3} \text{ feet.}$$

EXAMPLES. V.

[*The student should verify some of the following examples by an accurate figure and measurement.*]

1. The string (supposed straight) of a kite is 500 yards long and makes an angle whose sine is $\frac{5}{12}$ with the ground; find the height of the kite.

2. Assuming Snowdon to be 3600 feet high, find the distance (measured in a straight line) from its top to a point on the sea-level where the angle of elevation of the top is an angle whose sine is $\frac{3}{5}$.

3. At a distance of 770 feet from the base of the tower of Strasburg Cathedral the angle of elevation of the top of its spire is 31°; given that $\tan 31^\circ = \cdot 6$, find the height of the spire.

4. From the top of a cliff, 200 feet high, the angle of depression of a boat is found to be 30°; find the distance of the boat from the foot of the cliff.

5. A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60°; when he retires 40 feet from the bank he finds the angle to be 30°; find the height of the tree and the breadth of the river.

6. At a certain point the angle of elevation of a tower is found to be such that its cotangent is $\frac{3}{5}$; on walking 32 feet directly toward the tower its angle of elevation is an angle whose cotangent is $\frac{2}{5}$. Find the height of the tower.

7. At a point A, the angle of elevation of a tower is found to be such that its tangent is $\frac{5}{12}$; on walking 240 feet nearer the tower the tangent of the angle of elevation is found to be $\frac{3}{4}$; what is the height of the tower?

8. Find the height of a chimney when it is found that, on walking towards it 100 feet in a horizontal line through its base, the angular elevation of its top changes from 30° to 45°.

9. An observer on the top of a cliff, 200 feet above the sea-level, observes the angles of depression of two ships at anchor to be 45° and 30° respectively; find the distances between the ships if the line joining them points to the base of the cliff.

10. From the top of a cliff an observer finds that the angles of depression of two buoys in the sea are 39° and 26° respectively; the buoys are 300 yards apart and the line joining them points straight at the foot of the cliff; find the height of the cliff and the distance of the nearest buoy from the foot of the cliff, given that $\cot 26° = 2{\cdot}0503$, and $\cot 39° = 1{\cdot}2349$.

11. A tower is 100 feet high; from a point on the horizontal plane through its base where it subtends an angle of 45°, the angle of elevation of the top of a vertical flagstaff on the top of the tower is found to be 52°; the tables give $\tan 52° = 1{\cdot}28$; what is the height of the flagstaff?

12. The angle of elevation of the top of an unfinished tower at a point distant 120 feet from its base is 45°; how much higher must the tower be raised so that its angle of elevation at the same point may be 60°?

13. The angle of elevation of the top of a tower is observed to be 60°; at a point 40 feet above the first point of observation the elevation is found to be 45°; find the height of the tower and its horizontal distance from the points of observation.

14. The upper part of a tree broken over by the wind makes an angle of 30° with the ground, and the distance from the root to the point where the top of the tree touches the ground is 50 feet; what was the height of the tree?

15. The horizontal distance between two towers is 60 feet and the angular depression of the top of the first as seen from the top of the second, which is 150 feet high, is 30°; find the height of the first.

16. Two pillars of equal height stand on either side of a roadway which is 100 feet wide; at a point in the roadway between the pillars the elevations of the tops of the pillars are 60° and 30°; find their height and the position of the point.

17. What is the angle of elevation of the sun when the length of the shadow of a pole is $\sqrt{3}$ times the height of the pole?

18. The shadow of a tower standing on a level plane is found to be 60 feet longer when the sun's altitude is 30° than when it is 45°. Prove that the height of the tower is $30(1+\sqrt{3})$ feet.

19. On a straight coast there are three objects A, B, and C such that $AB = BC = 2$ miles. A vessel approaches B in a line perpendicular to the coast and at a certain point AC is found to subtend an angle of 60°; after sailing in the same direction for ten minutes AC is found to subtend 120°; find the rate at which the ship is going.

CHAPTER IV.

APPLICATION OF ALGEBRAIC SIGNS TO TRIGONOMETRY.

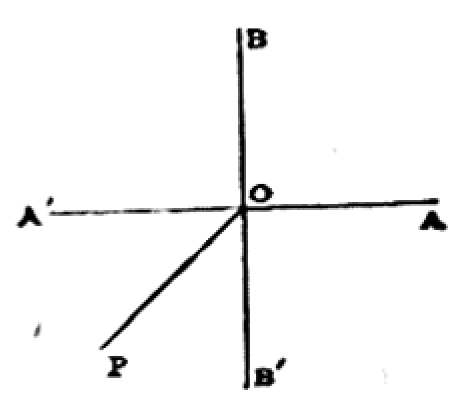
31. Positive and Negative Angles. In Art. 4, in treating of angles of any size, we spoke of the revolving line as if it always revolved in a direction opposite to that in which the hands of a watch revolve, when the watch is held with its face uppermost.

This direction is called counter-clockwise, thus ↶.

When the revolving line turns in this manner it is said to revolve in the positive direction and to trace out a positive angle.

When the line OP revolves in the opposite direction, *i.e.* in the same direction as the hands of the watch, it is said to revolve in the negative direction and to trace out a negative angle. This negative direction is clockwise; thus ↷.

32. Let the revolving line start from OA and revolve until it reaches a position OP, which lies between OA' and OB' and which bisects the angle $A'OB'$.



If it has revolved into the positive direction, it has traced out the positive angle whose measure is $+225°$.

If it has revolved in the negative direction, *i.e.* through OB', it has traced out the negative angle $-135°$.

Again, suppose we only know that the revolving line is in the above position. It may have made one, two, three... complete revolutions and then have described the positive angle $+225°$. Or again, it may have made one, two, three... complete revolutions in the negative direction and then have described the negative angle $-135°$.

In the first case, the angle it has described is either $225°$, or $360° + 225°$, or $2 \times 360° + 225°$, or $3 \times 360° + 225°$ ...... *i.e.* $225°$, or $585°$, or $945°$, or $1305°$....

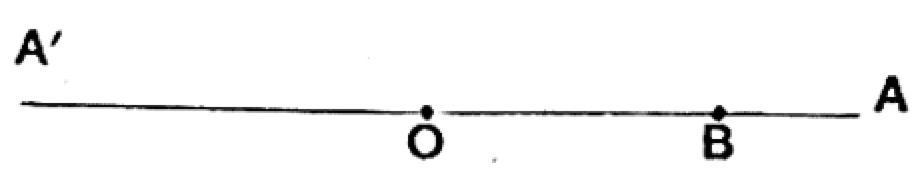
In the second case, the angle it has described is $-135°$, or $-360° - 135°$, or $-2 \times 360° - 135°$, or $-3 \times 360° - 135°$ ...... *i.e.* $-135°$, or $-495°$, or $-855°$, or $-1215°$....

33. Positive and Negative Lines. Suppose that a man is told to start from a given milestone on a straight road and to walk 1000 yards along the road and then to stop. Unless we are told the *direction* in which he started, we do not know his position when he stops. All we know is that he is either at a distance 1000 yards on one side of the milestone or at the same distance on the other side.

In measuring distances along a straight line it is therefore convenient to have a standard direction; this direction is called the positive direction and all distances measured along it are said to be positive. The opposite direction is called the negative direction, and all distances measured along it are said to be negative.

The standard, or positive, directions for lines drawn parallel to the foot of the page is towards the right.

The length OA is in the positive direction. The length OA' is in the negative direction. If the magnitude of the distance OA or OA' be a, the point A is at a distance $+a$ from O and the point A' is at a distance $-a$ from O.



All lines measured to the right have then the positive sign prefixed; all lines to the left have the negative sign prefixed.

If a point start from O and describe a positive distance OA, and then a distance AB back again toward O, equal numerically to b, the total distance it has described measured in the positive direction is $OA + AB$,

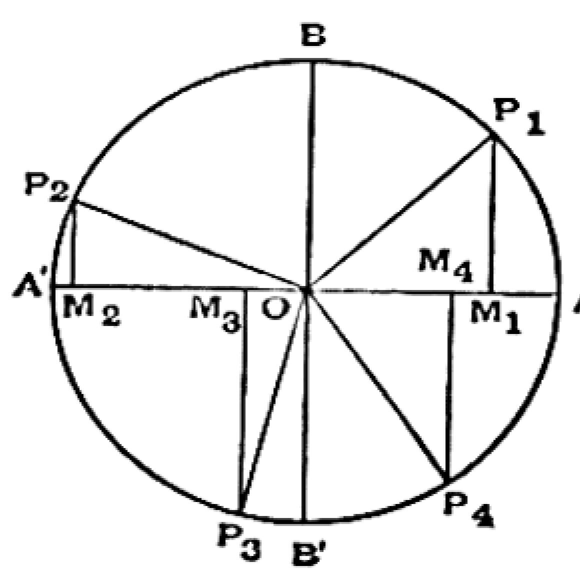
$$i.e. \ +a+(-b), \ i.e. \ a-b.$$

34. For lines at right angles to AA', the positive direction is from O towards the top of the page, *i.e.* the direction of OB (Fig. Art. 32). All lines measured from O towards the foot of the page, *i.e.* in the direction OB', are negative.

35. *Trigonometrical ratios for an angle of any magnitude.*

Let OA be the initial line (drawn in the positive direction) and let OA' be drawn in the opposite direction to OA.

Let BOB' be a line at right angles to OA, its positive direction being OB.



Let a revolving line OP start from OA and revolving in either direction, positive or negative, trace out an angle of any magnitude whatever. From a point P in the revolving line draw PM perpendicular to AOA'.

[Four positions of the revolving line are given in the figure, one in each of the four quadrants, and the suffixes 1, 2, 3 and 4 are attached to P for the purpose of distinction.]

We then have the following definitions, which are the

same as those given in Art. 7 for the simple case of an acute angle:

$\frac{MP}{OP}$ is called the **Sine** of the angle AOP,

$\frac{OM}{OP}$ „ „ **Cosine** „ „

$\frac{MP}{OM}$ „ „ **Tangent** „ „

$\frac{OM}{MP}$ „ „ **Cotangent** „ „

$\frac{OP}{OM}$ „ „ **Secant** „ „

$\frac{OP}{MP}$ „ „ **Cosecant** „ „

The quantities $1-\cos AOP$, and $1-\sin AOP$ are respectively called the **Versed Sine** and the **Coversed Sine** of AOP.

36. In exactly the same manner as in Art. 17 it may be shewn that, for all values of the angle $AOP\ (=\theta)$, we have

$$\sin^2\theta+\cos^2\theta=1,$$

$$\frac{\sin\theta}{\cos\theta}=\tan\theta,$$

$$\sec^2\theta=1+\tan^2\theta,$$

and $$\operatorname{cosec}^2\theta=1+\cot^2\theta.$$

37. Signs of the trigonometrical ratios.

First quadrant. Let the revolving line be in the first quadrant, as OP_1. This revolving line is always positive.

Here OM_1 and M_1P_1 are both positive, so that all the trigonometrical ratios are then positive.

Second quadrant. Let the revolving line be in the second quadrant, as OP_2. Here M_2P_2 is positive and OM_2 is negative.

The sine, being equal to the ratio of a positive quantity to a positive quantity, is therefore positive.

The cosine, being equal to the ratio of a negative quantity to a positive quantity, is therefore negative.

The tangent, being equal to the ratio of a positive quantity to a negative quantity, is therefore negative.

The cotangent is negative.

The cosecant is positive.

The secant is negative.

Third quadrant. If the revolving line be, as OP_3, in the third quadrant, we have both M_3P_3 and OM_3 negative

The sine is therefore negative.

The cosine is negative.

The tangent is positive.

The cotangent is positive.

The cosecant is negative.

The secant is negative.

Fourth quadrant. Let the revolving line be in the fourth quadrant, as OP_4. Here M_4P_4 is negative and OM_4 is positive.

The sine is therefore negative.

The cosine is positive.

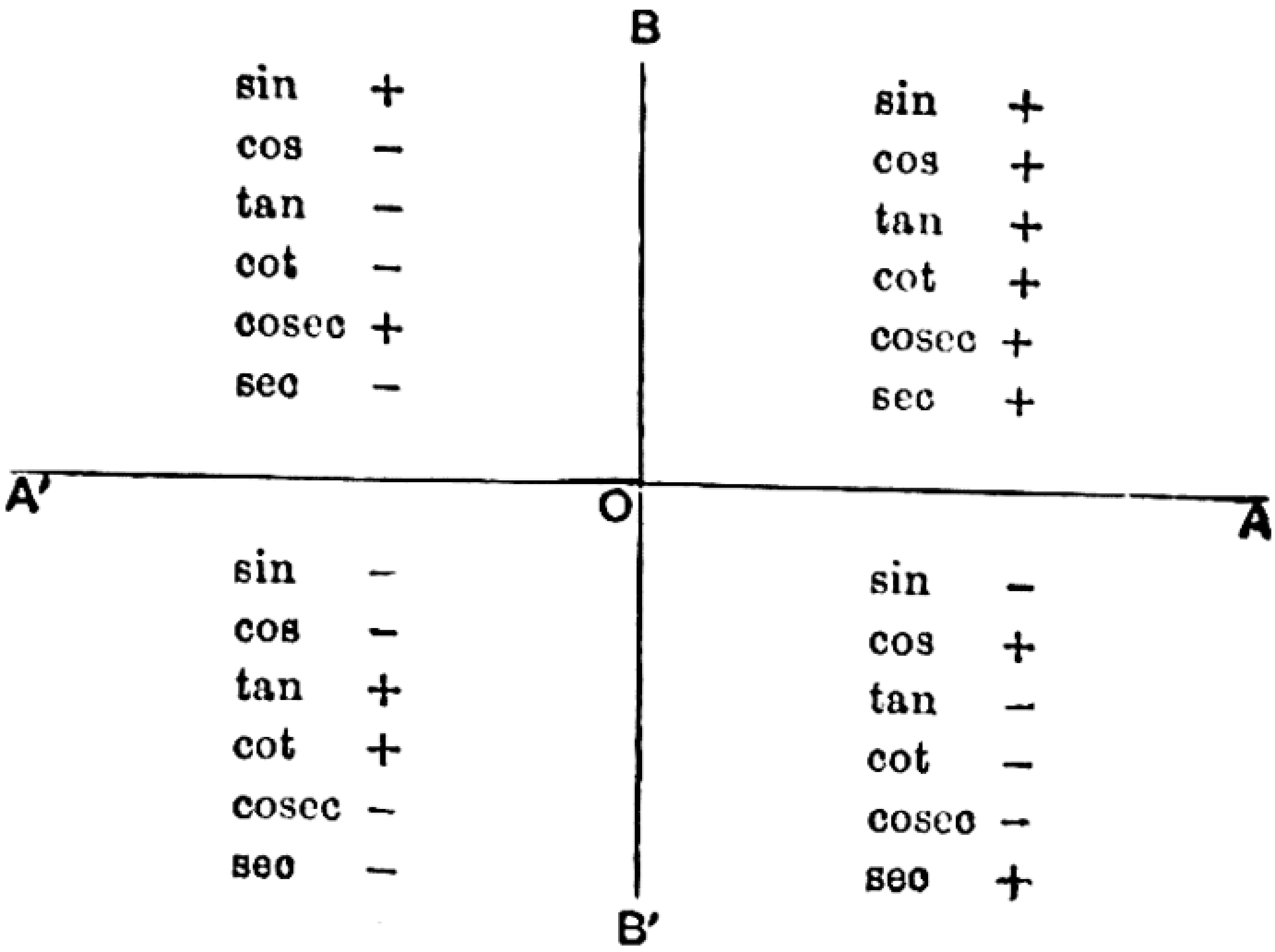
The tangent is negative.

The cotangent is negative.

The cosecant is negative.

The secant is positive.

The annexed table shews the signs of the trigonometrical ratios according to the quadrant in which lies the revolving line, which bounds the angle considered.



MISCELLANEOUS EXAMPLES. VI.

1. Draw the angle whose sine is $\frac{2}{3}$ and the angle whose tangent is $\frac{2}{3}$; measure them with a protractor.

2. Construct the angle whose sine is $\frac{3}{5}$ and the angle whose cosine is $\frac{5}{13}$; find by measurement the sine of the sum of the two angles.

3. A balloon starts vertically at a distance of 1000 feet from the observer; when it has risen a certain height he observes its angle of elevation to be 45°, and at a certain time later he finds it to be 58°; given that $\tan 58° = 1{\cdot}6$, find how far the balloon had risen between the two observations.

4. If $\sin\theta$ equal $\dfrac{x^2-y^2}{x^2+y^2}$, find the values of $\cos\theta$ and $\cot\theta$.

5. If
$$\sin\theta = \frac{m^2+2mn}{m^2+2mn+2n^2},$$
prove that
$$\tan\theta = \frac{m^2+2mn}{2mn+2n^2}.$$

6. If
$$\cos\theta - \sin\theta = \sqrt{2}\sin\theta,$$
prove that
$$\cos\theta + \sin\theta = \sqrt{2}\cos\theta.$$

7. Prove that

$$\operatorname{cosec}^6 a - \cot^6 a = 3 \operatorname{cosec}^2 a \cot^2 a + 1.$$

8. Express $2 \sec^2 A - \sec^4 A - 2 \operatorname{cosec}^2 A + \operatorname{cosec}^4 A$ in terms of $\tan A$.

9. Solve the equation $3 \operatorname{cosec}^2 \theta = 2 \sec \theta$.

10. A man on a cliff observes a boat at an angle of depression of 30°, which is making for the shore immediately beneath him. Three minutes later the angle of depression of the boat is 60°. How soon will it reach the shore?

11. Prove that the equation $\sin \theta = x + \dfrac{1}{x}$ is impossible if x be real.

12. Shew that the equation $\sec^2 \theta = \dfrac{4xy}{(x+y)^2}$ is only possible when $x = y$.

By drawing the angles as accurately as possible, and by measurement, find the values of the following, and compare the answers obtained with the values given on Pages 224—231:

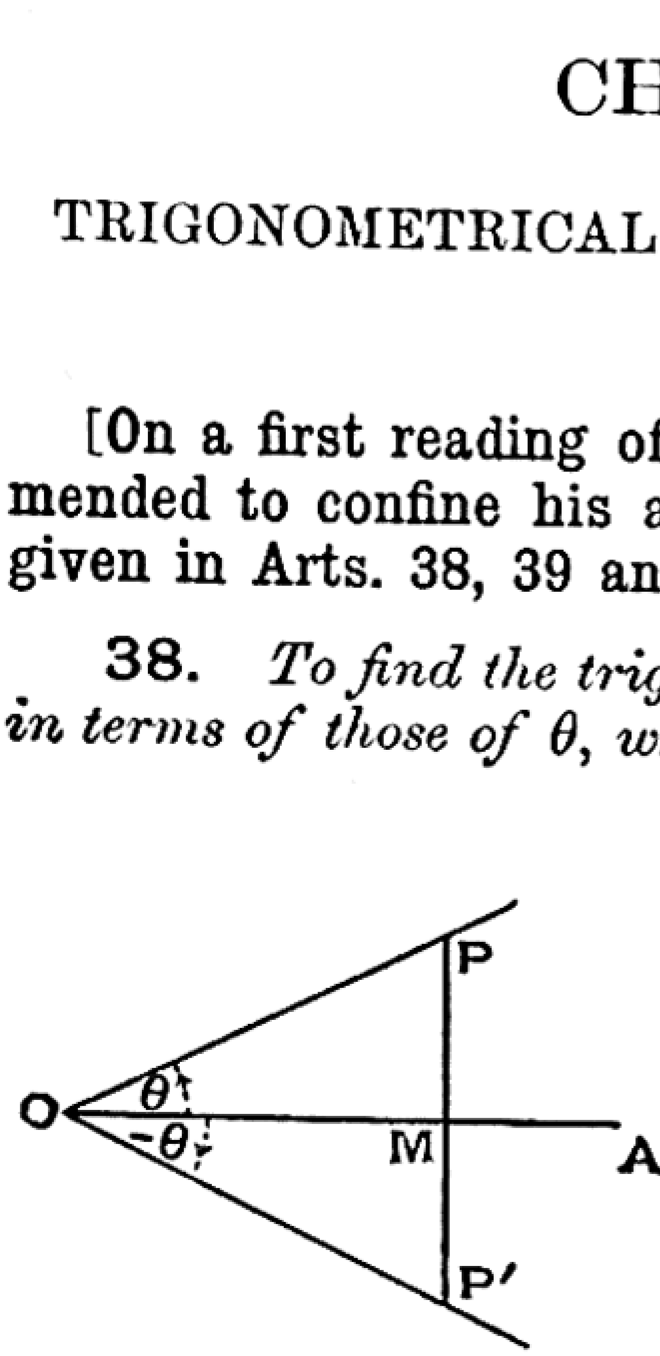
13. $\sin 29°$.
14. $\sin 71°$.
15. $\tan 42°$.
16. $\cos 43°$.
17. $\cot 49°$.
18. $\operatorname{cosec} 72°$.
19. $\sec 51°$.

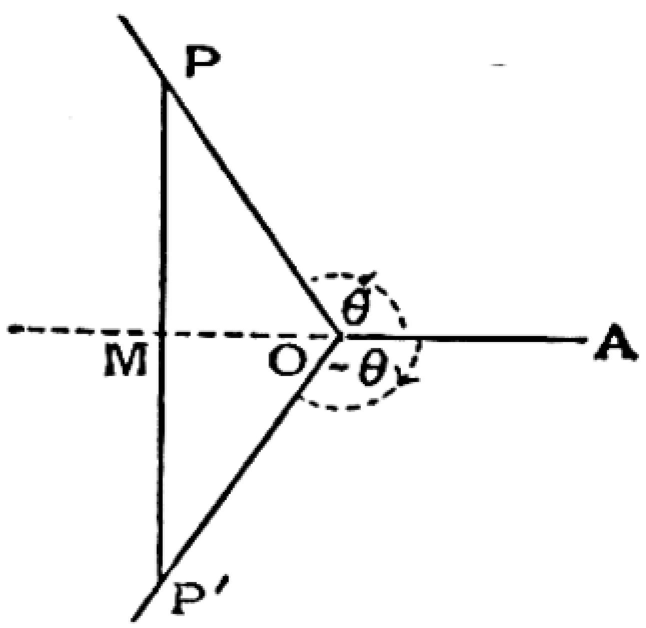
CHAPTER V.

TRIGONOMETRICAL FUNCTIONS OF CONNECTED ANGLES.

[On a first reading of the subject, the student is recommended to confine his attention to the first of the figures given in Arts. 38, 39 and 40.]

38. *To find the trigonometrical ratios of an angle* $(-\theta)$ *in terms of those of* θ, *when* θ *is less than two right angles.*





Let the revolving line, starting from OA, revolve through any angle θ and stop in the position OP.

Draw PM perpendicular to OA (or OA produced) and produce it to P', so that the lengths of PM and MP' are equal.

In the geometrical triangles MOP and MOP', we have the two sides OM and MP equal to the two OM and MP', and the included angles OMP and OMP' are right angles.

Hence, the magnitudes of the angles MOP and MOP' are the same, and OP is equal to OP'.

In each of the figures, the magnitudes of the angle AOP (measured ↺) and of the angle AOP' (measured ↻) are the same.

Hence the angle AOP' (measured ↻) is denoted by $-\theta$.

Also MP and MP' are equal in magnitude but are opposite in sign. (Art. 34.) We have therefore

$$\sin(-\theta) = \frac{MP'}{OP'} = \frac{-MP}{OP} = -\sin\theta,$$

$$\cos(-\theta) = \frac{OM}{OP'} = \frac{OM}{OP} = \cos\theta,$$

$$\tan(-\theta) = \frac{MP'}{OM} = \frac{-MP}{OM} = -\tan\theta,$$

$$\cot(-\theta) = \frac{OM}{MP'} = \frac{OM}{-MP} = -\cot\theta,$$

$$\operatorname{cosec}(-\theta) = \frac{OP'}{MP'} = \frac{OP}{-MP} = -\operatorname{cosec}\theta,$$

and
$$\sec(-\theta) = \frac{OP'}{OM} = \frac{OP}{OM} = \sec\theta.$$

[In this article, and the following articles, the values of the last four trigonometrical ratios may be found, without reference to the figure, from the values of the first two ratios.

Thus
$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta,$$

$$\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta,$$

$$\operatorname{cosec}(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\operatorname{cosec}\theta,$$

and
$$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta.]$$

Exs. $\sin(-30°) = -\sin 30° = -\frac{1}{2}$,

$\tan(-60°) = -\tan 60° = -\sqrt{3}$,

and $\cos(-45°) = \cos 45° = \frac{1}{\sqrt{2}}$.

39. *To find the trigonometrical ratios of the angle $(90° - \theta)$ in terms of those of θ, when θ is less than two right angles.*

The relations have already been discussed in Art. 24, for values of θ less than a right angle.

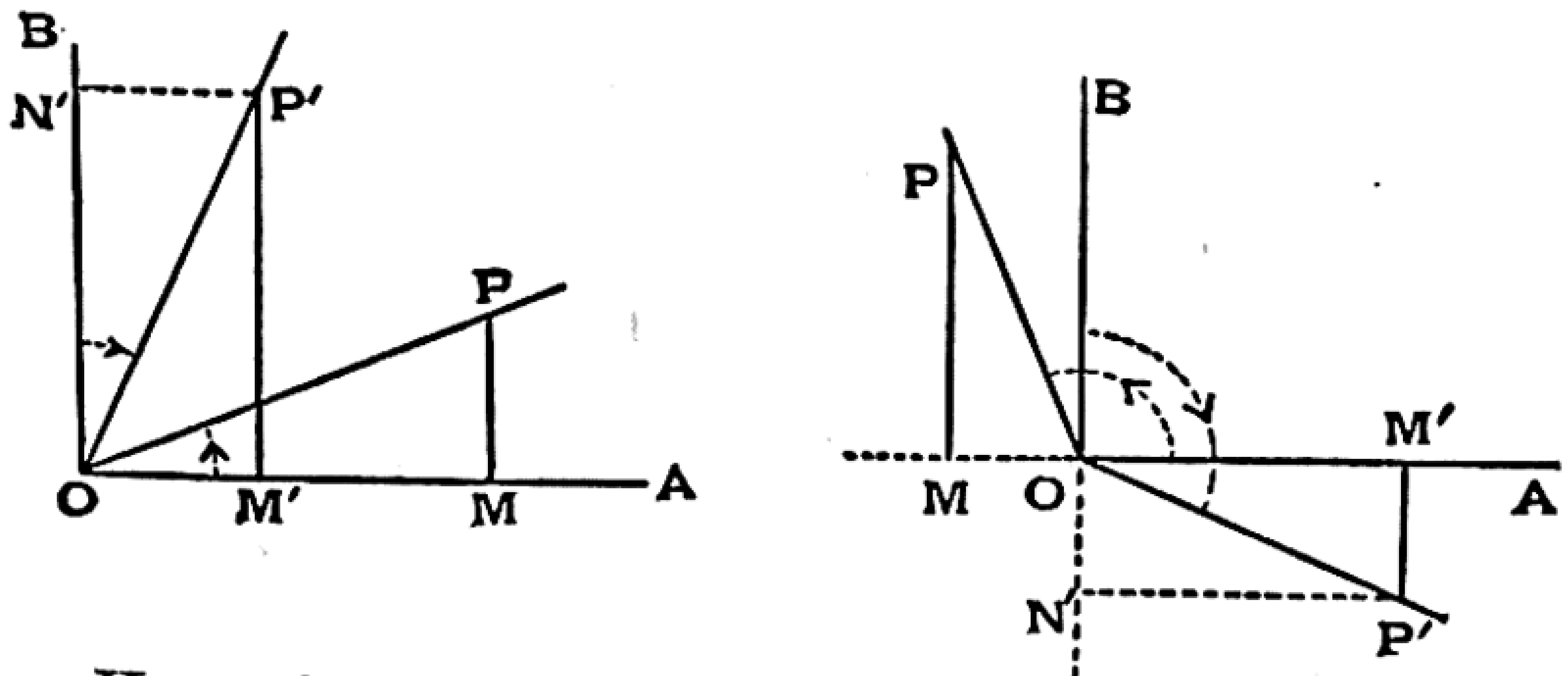
Let the revolving line, starting from OA, trace out any angle AOP denoted by θ.

To obtain the angle $90° - \theta$, let the revolving line rotate to B and then rotate from B in the opposite direction through the angle θ, and let the position of the revolving line be then OP'.

The angle AOP' is then $90° - \theta$.

Take OP' equal to OP, and draw $P'M'$ and PM perpendicular to OA, produced if necessary. Also draw $P'N'$ perpendicular to OB, produced if necessary.

In each figure, the angles AOP and BOP' are numerically equal, by construction.



Hence, in each figure,

$$\angle MOP = \angle N'OP' = \angle OP'M',$$

since ON' and $M'P'$ are parallel.

[For, in the second figure,

$\angle MOP = 2$ rt. $\angle^s - \angle AOP = 2$ rt. $\angle^s - \angle P'OB = \angle N'OP'$.]

Hence the triangles MOP and $M'P'O$ are equal in all respects, and therefore $OM = M'P'$ numerically,

and $$OM' = MP \text{ numerically.}$$

Also, in each figure, OM and $M'P'$ are of the same sign, and so also are MP and OM',

$$\textit{i.e. } OM = +M'P', \text{ and } OM' = +MP.$$

Hence

$$\mathbf{sin}\,(\mathbf{90}^\circ - \boldsymbol{\theta}) = \sin AOP' = \frac{M'P'}{OP'} = \frac{OM}{OP} = \mathbf{cos}\,\boldsymbol{\theta},$$

$$\mathbf{cos}\,(\mathbf{90}^\circ - \boldsymbol{\theta}) = \cos AOP' = \frac{OM'}{OP'} = \frac{MP}{OP} = \mathbf{sin}\,\boldsymbol{\theta},$$

$$\tan(90^\circ - \theta) = \tan AOP' = \frac{M'P'}{OM'} = \frac{OM}{MP} = \cot\theta,$$

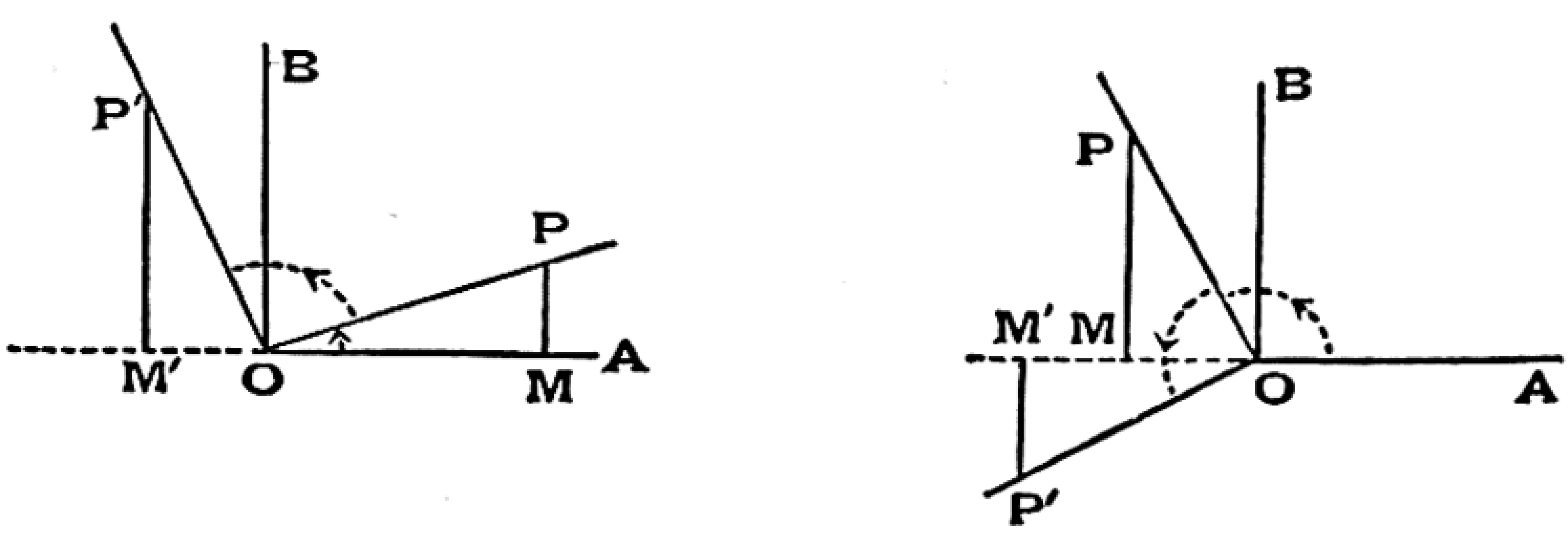
$$\cot(90^\circ - \theta) = \cot AOP' = \frac{OM'}{M'P'} = \frac{MP}{OM} = \tan\theta,$$

$$\sec(90^\circ - \theta) = \sec AOP' = \frac{OP'}{OM'} = \frac{OP}{MP} = \operatorname{cosec}\theta,$$

and

$$\operatorname{cosec}(90^\circ - \theta) = \operatorname{cosec} AOP' = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\theta.$$

40. *To find the trigonometrical ratios of the angle* $(90^\circ + \theta)$ *in terms of those of* θ, *when* θ *is less than two right angles.*



Let the revolving line, starting from OA, trace out any angle θ and let OP be the position of the revolving line then, so that the angle AOP is θ.

Let the revolving line turn through a right angle from OP in the positive direction to the position OP', so that the angle AOP' is $(90° + \theta)$.

Take OP' equal to OP and draw PM and $P'M'$ perpendicular to AO, produced if necessary. In each figure, since POP' is a right angle, the sum of the angles MOP and $P'OM'$ is always a right angle.

Hence $\angle MOP = 90° - \angle P'OM' = \angle OP'M'$.

The two triangles MOP and $M'P'O$ are therefore equal in all respects.

Hence OM and $M'P'$ are numerically equal, as also MP and OM' are numerically equal.

In each figure, OM and $M'P'$ have the same sign, whilst MP and OM' have the opposite sign, so that

$$M'P' = +OM, \text{ and } OM' = -MP.$$

We therefore have

$$\sin(90° + \theta) = \sin AOP' = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\theta,$$

$$\cos(90° + \theta) = \cos AOP' = \frac{OM'}{OP'} = \frac{-MP}{OP} = -\sin\theta,$$

$$\tan(90° + \theta) = \tan AOP' = \frac{M'P'}{OM'} = \frac{OM}{-MP} = -\cot\theta,$$

$$\cot(90° + \theta) = \cot AOP' = \frac{OM'}{M'P'} = \frac{-MP}{OM} = -\tan\theta,$$

$$\sec(90° + \theta) = \sec AOP' = \frac{OP'}{OM'} = \frac{OP}{-MP} = -\operatorname{cosec}\theta,$$

and

$$\operatorname{cosec}(90° + \theta) = \operatorname{cosec} AOP' = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\theta.$$

Exs. $\sin 150° = \sin(90° + 60°) = \cos 60° = \frac{1}{2}$,

$\cos 135° = \cos(90° + 45°) = -\sin 45° = -\frac{1}{\sqrt{2}}$,

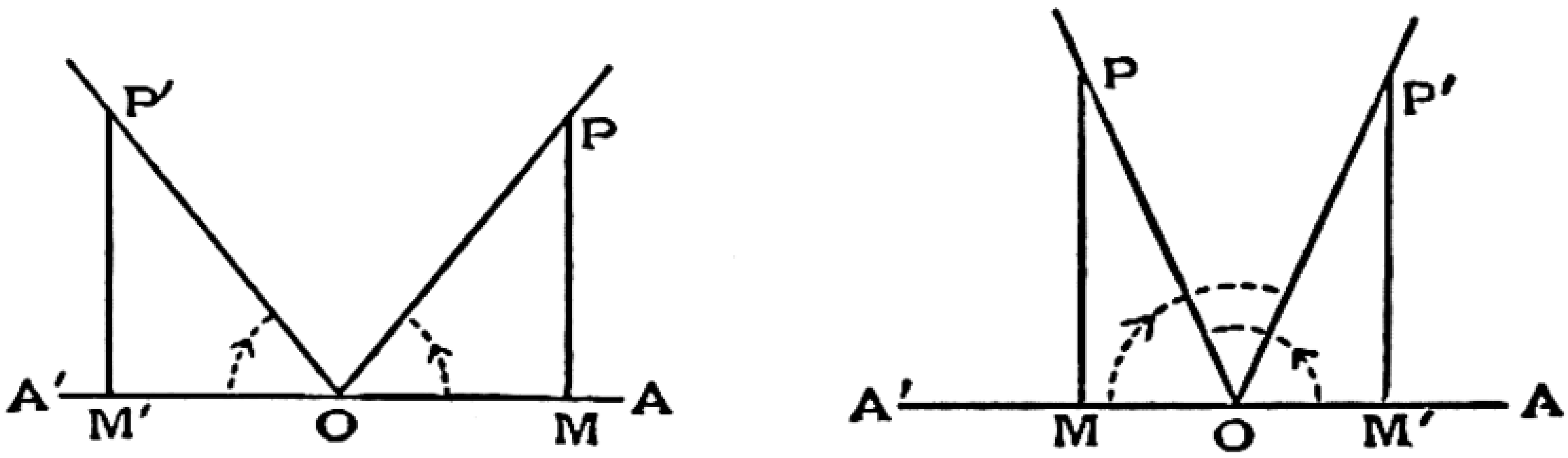
and $\tan 120° = \tan(90° + 30°) = -\cot 30° = -\sqrt{3}$.

41. Supplementary Angles.

Two angles are said to be supplementary when their sum is equal to two right angles, *i.e.* the supplement of any angle θ is $180° - \theta$.

Exs. The supplement of $30° = 180° - 30° = 150°$.
The supplement of $120° = 180° - 120° = 60°$.

42. *To find the values of the trigonometrical ratios of the angle* $(180° - \theta)$ *in terms of those of the angle* θ, *when* θ *is less than two right angles.*



Let the revolving line start from OA and describe any angle $AOP \, (= \theta)$.

To obtain the angle $180° - \theta$, let the revolving line start from OA and, after revolving through two right angles (*i.e.* into the position OA'), then revolve back through an angle θ into the position OP'. The angle $A'OP'$ is then equal in magnitude but opposite in sign to the angle AOP.

Also the angle AOP' is $180° - \theta$.

Take OP' equal to OP, and draw $P'M'$ and PM perpendicular to AOA'.

The angles MOP and $M'OP'$ are equal, and hence the triangles MOP and $M'OP'$ are equal in all respects.

Hence OM and OM' are equal in magnitude, and so also are MP and $M'P'$.

In each figure, OM and OM' are drawn in opposite directions, whilst MP and $M'P'$ are drawn in the same direction, so that

$$OM' = -OM, \text{ and } M'P' = +MP.$$

CHAPTER VI.

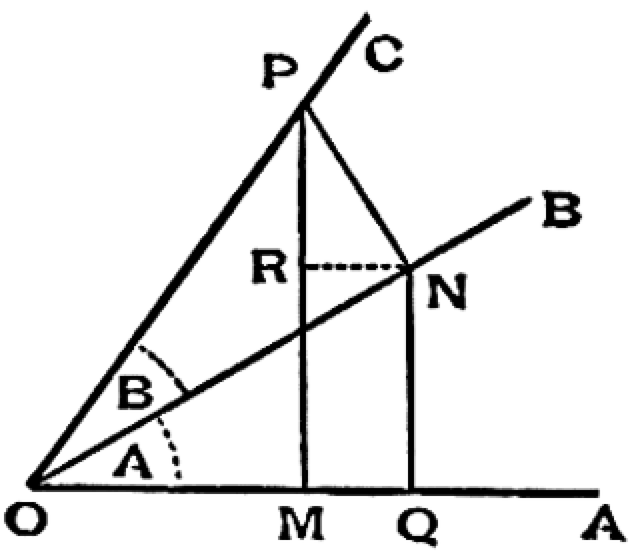
TRIGONOMETRICAL RATIOS OF THE SUM AND DIFFERENCE OF TWO ANGLES.

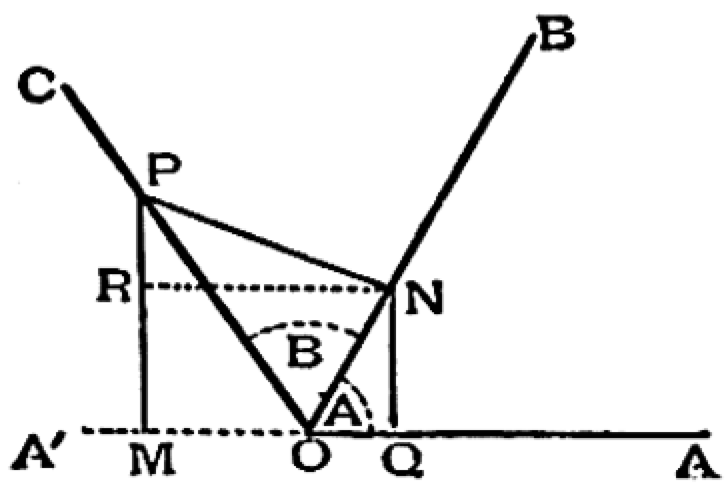
[On a first reading of the subject, the student is recommended to confine his attention to the first of the figures given in Art. 44.]

44. Theorem. *To prove that*

$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$

and $$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$





Let the revolving line start from OA and trace out the angle AOB ($=A$), and then trace out the further angle BOC ($=B$).

In the final position of the revolving line take any point P, and draw PM and PN perpendicular to OA and OB respectively; through N draw NR parallel to AO to meet MP in R, and draw NQ perpendicular to OA.

The angle

$$RPN = 90^\circ - \angle PNR = \angle RNO = \angle NOQ = A.$$

Hence $\sin(A+B) = \sin AOP = \frac{MP}{OP} = \frac{MR+RP}{OP}$

$$= \frac{QN}{OP} + \frac{RP}{OP} = \frac{QN}{ON}\frac{ON}{OP} + \frac{RP}{NP}\frac{NP}{OP}$$

$$= \sin A \cos B + \cos RPN \sin B.$$

$$\therefore \mathbf{\sin(A+B) = \sin A \cos B + \cos A \sin B}.$$

Again $\cos(A+B) = \cos AOP = \frac{OM}{OP} = \frac{OQ-MQ}{OP}$

$$= \frac{OQ}{OP} - \frac{RN}{OP} = \frac{OQ}{ON}\frac{ON}{OP} - \frac{RN}{NP}\frac{NP}{OP}$$

$$= \cos A \cos B - \sin RPN \sin B.$$

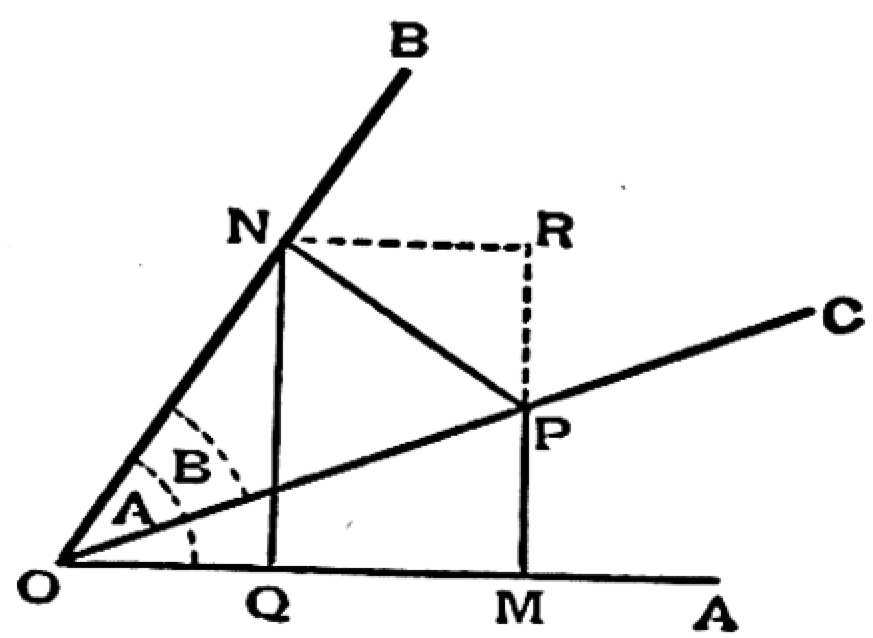
$$\therefore \mathbf{\cos(A+B) = \cos A \cos B - \sin A \sin B}.$$

45. Theorem. *To prove that*

$$\sin(A-B) = \sin A \cos B - \cos A \sin B,$$

and $$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

Let the revolving line starting from the initial line OA trace out the angle $AOB\,(=A)$, and then, revolving in the opposite direction, trace out the angle BOC, whose magnitude is B. The angle AOC is therefore $A-B$.



Take a point P in the final position of the revolving line, and draw PM and PN perpendicular to OA and OB respectively; from N draw NQ and NR perpendicular to OA and MP respectively.

The angle $RPN = 90° - \angle PNR = \angle RNB = \angle QON = A$. Hence

$$\sin(A-B) = \sin AOC = \frac{MP}{OP} = \frac{MR - PR}{OP} = \frac{QN}{OP} - \frac{PR}{OP}$$

$$= \frac{QN}{ON}\frac{ON}{OP} - \frac{PR}{PN}\frac{PN}{OP}$$

$$= \sin A \cos B - \cos RPN \sin B,$$

so that $\mathbf{\sin(A-B) = \sin A \cos B - \cos A \sin B}$.

Also $\cos(A-B) = \frac{OM}{OP} = \frac{OQ + QM}{OP} = \frac{OQ}{OP} + \frac{NR}{OP}$

$$= \frac{OQ}{ON}\frac{ON}{OP} + \frac{NR}{NP}\frac{NP}{OP} = \cos A \cos B + \sin NPR \sin B,$$

so that $\mathbf{\cos(A-B) = \cos A \cos B + \sin A \sin B}$.

46. The figures of the two preceding articles are drawn for the cases when A and B are both acute angles; we shall for the present assume that the theorems hold for all angles; proofs will be given in a later chapter.

These theorems, which give respectively the trigonometrical functions of the sum and differences of two angles in terms of the functions of the angles themselves, are often called the **Addition and Subtraction Theorems.**

47. Ex. 1. *Find the values of* $\sin 75°$ *and* $\cos 75°$.

$$\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}},$$

and

$$\cos 75° = \cos(45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Ex. 2. *If A and B be acute angles and if* $\cos A = \frac{1}{3}$ *and* $\sin B = \frac{4}{5}$ *find the values of* $\sin(A+B)$ *and* $\cos(A-B)$. *Verify the results, to the second place of decimals, by a drawing and measurement.*

$$\cos A = \frac{1}{3}; \quad \therefore \ \sin A = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}.$$

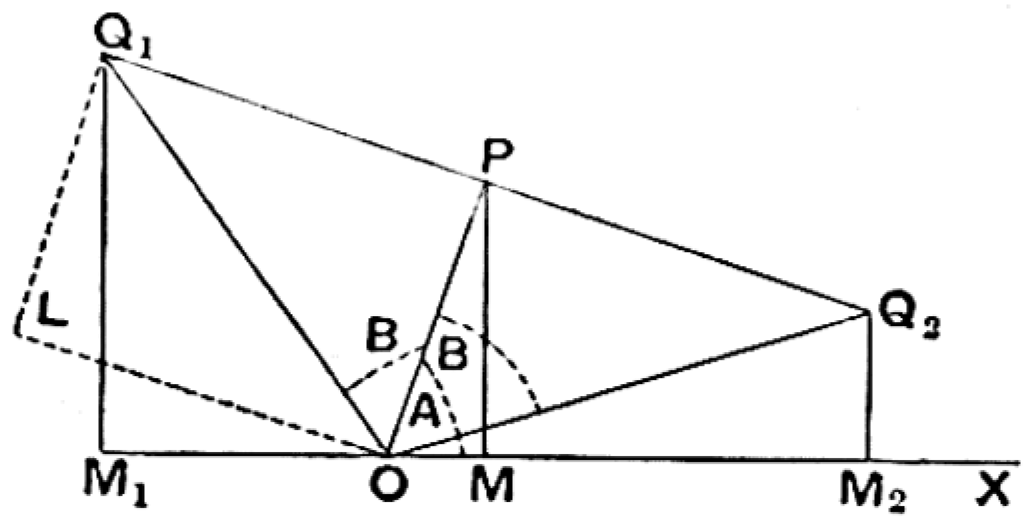
$$\sin B = \frac{4}{5}; \quad \therefore \ \cos B = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}.$$

$$\therefore \ \sin(A+B) = \frac{2\sqrt{2}}{3} \times \frac{3}{5} + \frac{1}{3} \times \frac{4}{5} = \frac{6\sqrt{2}+4}{15} = \frac{6 \times 1{\cdot}4142\ldots + 4}{15}$$

$$= \frac{12{\cdot}4852}{15} = {\cdot}8323\ldots.$$

$$\cos(A-B) = \frac{1}{3} \times \frac{3}{5} + \frac{2\sqrt{2}}{3} \times \frac{4}{5} = \frac{3+8\sqrt{2}}{15} = \frac{14{\cdot}3136\ldots}{15}$$

$$= {\cdot}9542\ldots$$



To construct the figure; Along OX mark off OM equal to one inch and erect a perpendicular MP; with centre O and radius 3 inches describe a circle to cut MP in P; then XOP is the $\angle A$. Draw OL perpendicular to OP and equal to 4 inches and draw LQ_1 parallel to OP; with centre O and radius 5 inches describe a circle to cut LQ_1 in Q_1; then the perpendicular from Q_1 on OP will be found to meet it in P and POQ_1 is the angle B. Produce Q_1P to Q_2 making PQ_2 equal to Q_1P; then POQ_2 is the angle B also.

Hence $\angle XOQ_1$ is $A+B$ and $\angle XOQ_2$ is $A-B$.

Draw Q_1M_1, Q_2M_2 perpendicular to OX.

On measurement, we obtain

$$Q_1M_1 = 4{\cdot}16 \text{ inches and } OM_2 = 4{\cdot}77 \text{ inches.}$$

$$\therefore \ \sin(A+B) = \frac{Q_1M_1}{OQ_1} = \frac{4{\cdot}16}{5} = {\cdot}83,$$

and

$$\cos(A-B) = \frac{OM_2}{OQ_2} = \frac{4{\cdot}77}{5} = {\cdot}95.$$

EXAMPLES. VIII.

1. If $\sin\alpha = \frac{4}{5}$ and $\cos\beta = \frac{8}{17}$, find the value of $\sin(\alpha+\beta)$.

2. If $\tan\alpha = \frac{4}{3}$ and $\tan\beta = \frac{7}{24}$, find the value of $\cos(\alpha-\beta)$.

3. If $\cos\alpha = \frac{4}{5}$ and $\sin\beta = \frac{5}{13}$, find the values of $\sin(\alpha-\beta)$ and $\cos(\alpha+\beta)$. Verify by a graph.

4. If $\sin\alpha = \frac{3}{5}$ and $\cos\beta = \frac{9}{41}$, find the values of $\sin(\alpha-\beta)$ and $\cos(\alpha+\beta)$. Verify by a graph and accurate measurement.

5. If $\sin\alpha = \frac{45}{53}$ and $\sin\beta = \frac{33}{65}$, find the values of $\sin(\alpha-\beta)$ and $\sin(\alpha+\beta)$.

6. If $\sin\alpha = \frac{15}{17}$ and $\cos\beta = \frac{12}{13}$, find the values of $\sin(\alpha+\beta)$, $\cos(\alpha-\beta)$, and $\tan(\alpha+\beta)$.

Prove that

7. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$.

8. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$.

9. $\cos(45^\circ - A)\cos(45^\circ - B) - \sin(45^\circ - A)\sin(45^\circ - B) = \sin(A+B)$.

10. $\sin(45^\circ + A)\cos(45^\circ - B) + \cos(45^\circ + A)\sin(45^\circ - B) = \cos(A-B)$.

11. $\dfrac{\sin(A-B)}{\cos A\cos B} + \dfrac{\sin(B-C)}{\cos B\cos C} + \dfrac{\sin(C-A)}{\cos C\cos A} = 0$.

12. $\sin 105^\circ + \cos 105^\circ = \cos 45^\circ$.

13. $\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ$.

14. $\cos\alpha\cos(\gamma-\alpha) - \sin\alpha\sin(\gamma-\alpha) = \cos\gamma$.

48. From Arts. 44 and 45, we have,

$$\sin(A+B) = \sin A\cos B + \cos A\sin B,$$

and

$$\sin(A-B) = \sin A\cos B - \cos A\sin B.$$

Hence, by addition and subtraction, we have

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B\ldots\ldots(1),$$

and

$$\sin(A+B) - \sin(A-B) = 2\cos A\sin B\ldots\ldots(2).$$

From the same articles we have,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

and $$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

Hence, by addition and subtraction, we have

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B \ldots\ldots(3),$$

and $$\cos(A-B) - \cos(A+B) = 2\sin A \sin B \ldots\ldots(4).$$

Put $A+B=C$, and $A-B=D$, so that

$$A = \frac{C+D}{2}, \text{ and } B = \frac{C-D}{2}.$$

On making these substitutions in the relations (1) to (4), and changing the signs of each side of the latter, we have

$$\mathbf{\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}} \ldots\ldots\mathbf{I},$$

$$\mathbf{\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}} \ldots\mathbf{II},$$

$$\mathbf{\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}} \ldots\mathbf{III},$$

and $$\mathbf{\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}} \ldots\mathbf{IV}.$$

[The student should carefully notice that the second factor of the right-hand member of IV is

$$\sin\frac{D-C}{2} \text{ and not } \sin\frac{C-D}{2}.]$$

****49. These relations I to IV are extremely important and should be very carefully committed to memory.**

On account of their great importance we give a geometrical proof for the case when *C* and *D* are acute angles.

Let *AOC* be the angle *C* and *AOD* the angle *D*. Bisect the angle *COD* by the straight line *OE*. On *OE* take a point *P* and draw *QPR* perpendicular to *OP* to meet *OC* and *OD* in *Q* and *R* respectively.

Draw *PL*, *QM*, and *RN* perpendicular to *OA*, and through *R* draw *RST* perpendicular to *PL* or *QM* to meet them in *S* and *T* respectively.

Since the angle *DOC* is $C-D$, each of the angles *DOE* and *EOC* is $\frac{C-D}{2}$, and also $\angle AOE = \angle AOD + \angle DOE = D + \frac{C-D}{2} = \frac{C+D}{2}$.

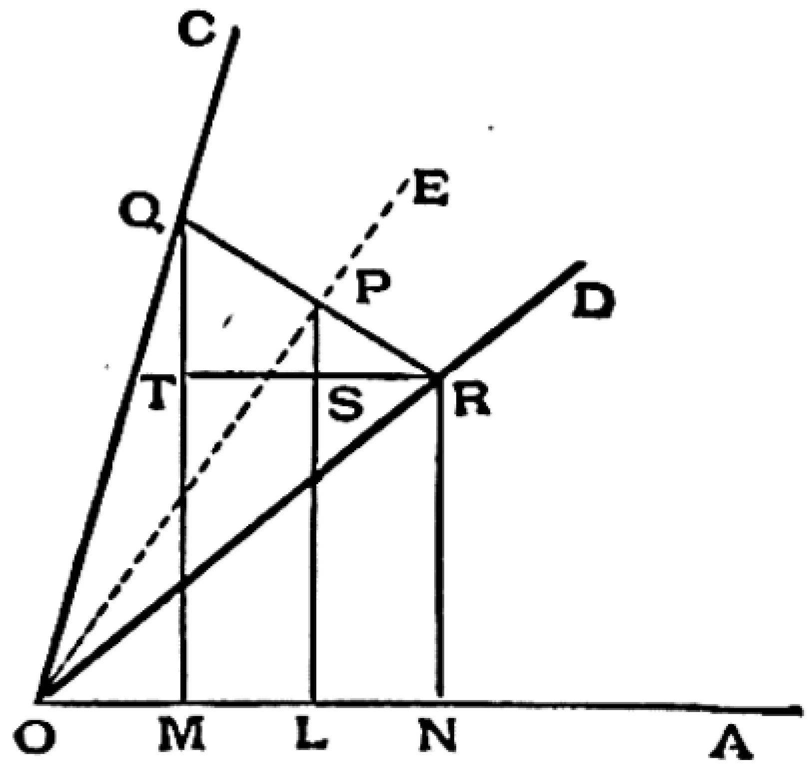
Since the two triangles POR and POQ are equal in all respects, we have $OQ=OR$, and $PR=PQ$, so that

$$RQ=2RP.$$

Hence $QT=2PS$, and $RT=2RS$, *i.e.* $MN=2LN=2ML$.

$$\therefore\ MQ+NR=TQ+2LS=2SP+2LS=2LP.$$

Also $$OM+ON=2OM+MN=2OM+2ML=2OL.$$



Hence $\sin C+\sin D=\dfrac{MQ}{OQ}+\dfrac{NR}{OR}=\dfrac{MQ+NR}{OR}$

$$=\frac{2LP}{OR}=2\frac{LP}{OP}\cdot\frac{OP}{OR}=2\sin LOP\cos POR$$

$$=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}.$$

Again,

$$\sin C-\sin D=\frac{MQ}{OQ}-\frac{NR}{OR}=\frac{MQ-NR}{OR}=\frac{TQ}{OR}$$

$$=2\frac{SP}{OR}=2\frac{SP}{RP}\cdot\frac{RP}{OR}=2\cos SPR\sin ROP$$

$$=2\cos\frac{C+D}{2}\sin\frac{C-D}{2},$$

$$\left[\text{for } \angle SPR=90^\circ-\angle SPO=\angle LOP=\frac{C+D}{2}\right].$$

Also, $$\cos C+\cos D=\frac{OM}{OQ}+\frac{ON}{OR}=\frac{OM+ON}{OR}$$

$$=2\frac{OL}{OR}=2\frac{OL}{OP}\frac{OP}{OR}$$

$$=2\cos LOP\cos POR=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}.$$

Finally, $$\cos D - \cos C = \frac{ON}{OR} - \frac{OM}{OQ} = \frac{ON - OM}{OR}$$

$$= \frac{MN}{OR} = 2\frac{SR}{OR} = \frac{2SR}{PR}\frac{PR}{OR}$$

$$= 2\sin SPR \,.\, \sin POR$$

$$= 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}.$$

50. The student is strongly urged to make himself perfectly familiar with the formulae of Art. 48 and to carefully practise himself in their application; perfect familiarity with these formulae will considerably facilitate his further progress.

The formulae are very useful, because they change sums and differences of certain quantities into products of certain other quantities, and products of quantities are easily dealt with by the help of logarithms.

We subjoin a few examples of their use.

Ex. 1. $\sin 6\theta + \sin 4\theta = 2\sin\frac{6\theta+4\theta}{2}\cos\frac{6\theta-4\theta}{2} = 2\sin 5\theta\cos\theta.$

Ex. 2. $\cos 3\theta - \cos 7\theta = 2\sin\frac{3\theta+7\theta}{2}\sin\frac{7\theta-3\theta}{2} = 2\sin 5\theta\sin 2\theta.$

Ex. 3. $$\frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \frac{2\cos\frac{75^\circ+15^\circ}{2}\sin\frac{75^\circ-15^\circ}{2}}{2\cos\frac{75^\circ+15^\circ}{2}\cos\frac{75^\circ-15^\circ}{2}}$$

$$= \frac{2\cos 45^\circ\sin 30^\circ}{2\cos 45^\circ\cos 30^\circ} = \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \cdot 57735\ldots\ldots$$

[This is an example of the simplification given by these formulae; it would be a very long and tiresome process to look out from the tables the values of sin 75°, sin 15°, cos 75°, and cos 15°, and then to perform the division of one long decimal fraction by another.]

Ex. 4. *Simplify the expression*

$$\frac{(\cos\theta - \cos 3\theta)(\sin 8\theta + \sin 2\theta)}{(\sin 5\theta - \sin\theta)(\cos 4\theta - \cos 6\theta)}.$$

On applying the formulae of Art. 48, this expression

$$=\frac{2\sin\frac{\theta+3\theta}{2}\sin\frac{3\theta-\theta}{2}\times 2\sin\frac{8\theta+2\theta}{2}\cos\frac{8\theta-2\theta}{2}}{2\cos\frac{5\theta+\theta}{2}\sin\frac{5\theta-\theta}{2}\times 2\sin\frac{4\theta+6\theta}{2}\sin\frac{6\theta-4\theta}{2}}$$

$$=\frac{4\,.\sin 2\theta\sin\theta\,.\sin 5\theta\cos 3\theta}{4\,.\cos 3\theta\sin 2\theta\,.\sin 5\theta\sin\theta}=1.$$

Ex. 5. Shew that

$$\sin\alpha+\sin\beta-\sin\gamma-\sin(\alpha+\beta-\gamma)=4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\alpha-\gamma}{2}.$$

The left hand

$$=\sin\alpha+\sin\beta-[\sin\gamma+\sin(\alpha+\beta-\gamma)]$$

$$=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}-2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta-2\gamma}{2} \qquad \text{[Art. 48]}$$

$$=2\sin\frac{\alpha+\beta}{2}\left[\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta-2\gamma}{2}\right]$$

$$=2\sin\frac{\alpha+\beta}{2}\times 2\sin\frac{\frac{\alpha-\beta}{2}+\frac{\alpha+\beta-2\gamma}{2}}{2}\sin\frac{\frac{\alpha+\beta-2\gamma}{2}-\frac{\alpha-\beta}{2}}{2} \qquad \text{[Art. 48]}$$

$$=2\sin\frac{\alpha+\beta}{2}\times 2\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2}.$$

EXAMPLES. IX.

Prove that

1. $\sin 43^\circ+\sin 17^\circ=\cos 13^\circ.$

2. $\cos 25^\circ-\cos 35^\circ=\sin 5^\circ.$

3. $\dfrac{\sin 7\theta-\sin 5\theta}{\cos 7\theta+\cos 5\theta}=\tan\theta.$

4. $\dfrac{\cos 6\theta-\cos 4\theta}{\sin 6\theta+\sin 4\theta}=-\tan\theta.$

5. $\dfrac{\sin A+\sin 3A}{\cos A+\cos 3A}=\tan 2A.$

6. $\dfrac{\sin 7A-\sin A}{\sin 8A-\sin 2A}=\cos 4A\sec 5A.$

7. $\dfrac{\cos 2B+\cos 2A}{\cos 2B-\cos 2A}=\cot(A+B)\cot(A-B).$

8. $\dfrac{\sin 2A+\sin 2B}{\sin 2A-\sin 2B}=\dfrac{\tan(A+B)}{\tan(A-B)}.$

9. $\dfrac{\sin A+\sin 2A}{\cos A-\cos 2A}=\cot\dfrac{A}{2}$.

10. $\dfrac{\sin 5A-\sin 3A}{\cos 3A+\cos 5A}=\tan A$.

11. $\dfrac{\cos 2B-\cos 2A}{\sin 2B+\sin 2A}=\tan(A-B)$.

12. $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan\dfrac{A+B}{2}$.

13. $\dfrac{\sin A-\sin B}{\cos B-\cos A}=\cot\dfrac{A+B}{2}$.

14. $\cos 2\alpha+\cos 4\alpha+\cos 6\alpha+\cos 8\alpha=4\cos\alpha\cos 2\alpha\cos 5\alpha$.

15. $\dfrac{\sin(4A-2B)+\sin(4B-2A)}{\cos(4A-2B)+\cos(4B-2A)}=\tan(A+B)$.

16. $\dfrac{\cos 3\theta+2\cos 5\theta+\cos 7\theta}{\cos\theta+2\cos 3\theta+\cos 5\theta}=\dfrac{\cos 5\theta}{\cos 3\theta}$.

17. $\dfrac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}=\tan 4A$.

18. $\dfrac{\sin(\theta+\phi)-2\sin\theta+\sin(\theta-\phi)}{\cos(\theta+\phi)-2\cos\theta+\cos(\theta-\phi)}=\tan\theta$.

19. $\dfrac{\sin A+2\sin 3A+\sin 5A}{\sin 3A+2\sin 5A+\sin 7A}=\dfrac{\sin 3A}{\sin 5A}$.

20. $\dfrac{\sin(A-C)+2\sin A+\sin(A+C)}{\sin(B-C)+2\sin B+\sin(B+C)}=\dfrac{\sin A}{\sin B}$.

21. $\dfrac{\sin\theta+\sin 2\theta+\sin 3\theta}{\cos\theta+\cos 2\theta+\cos 3\theta}=\tan 2\theta$.

22. $\dfrac{\sin A-\sin 5A+\sin 9A-\sin 13A}{\cos A-\cos 5A-\cos 9A+\cos 13A}=\cot 4A$.

23. $\cos 3A+\cos 5A+\cos 7A+\cos 15A=4\cos 4A\cos 5A\cos 6A$.

24. $\cos(-A+B+C)+\cos(A-B+C)+\cos(A+B-C)$
$$+\cos(A+B+C)=4\cos A\cos B\cos C.$$

25 $\sin\alpha+\sin 2\alpha+\sin 4\alpha+\sin 5\alpha=4\cos\dfrac{\alpha}{2}\cos\dfrac{3\alpha}{2}\sin 3\alpha$.

26. $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$
$$=4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}.$$

51. The formulae (1), (2), (3), and (4) of Art. 48 are also very important. They should be remembered in the form

$$\mathbf{2 \sin A \cos B = \sin (A+B) + \sin (A-B)} \ldots(1),$$
$$\mathbf{2 \cos A \sin B = \sin (A+B) - \sin (A-B)} \ldots(2),$$
$$\mathbf{2 \cos A \cos B = \cos (A+B) + \cos (A-B)} \ldots(3),$$
and $$\mathbf{2 \sin A \sin B = \cos (A-B) - \cos (A+B)} \ldots(4).$$

They may be looked upon as the converse of the formulae I—IV. of Art. 48.

Ex. 1. $2 \sin 3\theta \cos \theta = \sin 4\theta + \sin 2\theta$.

Ex. 2. $2 \sin 5\theta \sin 3\theta = \cos 2\theta - \cos 8\theta$.

Ex. 3. $2 \cos 11\theta \cos 2\theta = \cos 13\theta + \cos 9\theta$.

Ex. 4. Simplify

$$\frac{\sin 8\theta \cos \theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos \theta - \sin 3\theta \sin 4\theta}.$$

By the above formulae, the expression

$$= \frac{\frac{1}{2}[\sin 9\theta + \sin 7\theta] - \frac{1}{2}[\sin 9\theta + \sin 3\theta]}{\frac{1}{2}[\cos 3\theta + \cos \theta] - \frac{1}{2}[\cos \theta - \cos 7\theta]}$$

$$= \frac{\sin 7\theta - \sin 3\theta}{\cos 3\theta + \cos 7\theta}$$

$$= \frac{2 \cos 5\theta \sin 2\theta}{2 \cos 5\theta \cos 2\theta}, \text{ by the formulae of Art. 48,}$$

$$= \tan 2\theta.$$

[The student should carefully notice the artifice of first employing the formulae of this article and then, to obtain a further simplification, employing the *converse* formulae of Art. 48. This artifice is often successful in simplifications.]

EXAMPLES. X.

Express as a sum or difference the following:

1. $2 \sin 3\theta \cos 2\theta$.

2. $2 \cos 3\theta \sin \theta$.

3. $2 \cos 5\theta \cos 2\theta$.

4. $2 \cos 9\theta \sin 2\theta$.

5. $2 \sin 2\theta \sin \theta$.

6. $2 \sin 5\theta \sin 7\theta$.

7. $2\cos 7\theta \sin 5\theta$.

8. $2\cos 11\theta \cos 3\theta$.

9. $2\sin 54^\circ \sin 66^\circ$.

10. $2\sin\frac{7\theta}{2}\cos\frac{3\theta}{2}$.

11. $2\sin\frac{3\theta}{2}\cos\frac{5\theta}{2}$.

12. $2\cos\frac{\theta}{4}\cos\frac{7\theta}{4}$.

13. $2\sin\frac{\theta}{4}\sin\frac{5\theta}{4}$.

Prove that

14. $\cos 3\theta \sin\theta + \cos 6\theta \sin 2\theta = \cos 5\theta \sin 3\theta$.

15. $\sin\theta\sin 2\theta + \sin 3\theta \sin 6\theta = \sin 4\theta \sin 5\theta$.

16. $\sin\frac{\theta}{2}\sin\frac{7\theta}{2} + \sin\frac{3\theta}{2}\sin\frac{11\theta}{2} = \sin 2\theta \sin 5\theta$.

17. $\cos 2\theta \cos\frac{\theta}{2} - \cos 3\theta \cos\frac{9\theta}{2} = \sin 5\theta \sin\frac{5\theta}{2}$.

18. $\sin A \sin(A+2B) - \sin B \sin(B+2A) = \sin(A-B)\sin(A+B)$.

19. $(\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A = 0$.

20. $\dfrac{2\sin(A-C)\cos C - \sin(A-2C)}{2\sin(B-C)\cos C - \sin(B-2C)} = \dfrac{\sin A}{\sin B}$.

21. $\dfrac{\sin A \sin 2A + \sin 3A \sin 6A + \sin 4A \sin 13A}{\sin A \cos 2A + \sin 3A \cos 6A + \sin 4A \cos 13A} = \tan 9A$.

22. $\dfrac{\cos 2A \cos 3A - \cos 2A \cos 7A + \cos A \cos 10A}{\sin 4A \sin 3A - \sin 2A \sin 5A + \sin 4A \sin 7A} = \cot 6A \cot 5A$.

23. $\cos(36^\circ - A)\cos(36^\circ + A) + \cos(54^\circ + A)\cos(54^\circ - A) = \cos 2A$.

24. $\sin(45^\circ + A)\sin(45^\circ - A) = \frac{1}{2}\cos 2A$.

25. $\sin(\beta-\gamma)\cos(\alpha-\delta) + \sin(\gamma-\alpha)\cos(\beta-\delta)$
$+\sin(\alpha-\beta)\cos(\gamma-\delta) = 0$.

26. Express $4\cos\frac{\theta}{2}\cos\theta\cos\frac{5\theta}{2}$ as the sum of four cosines.

52. *To prove that* $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$, *and that* $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$.

By Art. 44 we have

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}},$$

by dividing both numerator and denominator by $\cos A\cos B$.

$$\therefore\ \mathbf{\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}}.$$

Again, by Art. 45,

$$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}$$

$$=\frac{\dfrac{\sin A}{\cos A}-\dfrac{\sin B}{\cos B}}{1+\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}},$$

by dividing as before.

$$\therefore\ \mathbf{\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}}.$$

**53. The formulae of the preceding article may be obtained geometrically from the figures of Arts. 44 and 45.

(1) Taking the figure of Art. 44 we have

$$\tan(A+B)=\frac{MP}{OM}=\frac{QN+RP}{OQ-RN}$$

$$=\frac{\dfrac{QN}{OQ}+\dfrac{RP}{OQ}}{1-\dfrac{RN}{OQ}}=\frac{\tan A+\dfrac{RP}{OQ}}{1-\dfrac{RN}{RP}\dfrac{RP}{OQ}}.$$

But, since the angles RPN and QON are equal, the triangles RPN and QON are similar, so that

$$\frac{RP}{PN}=\frac{OQ}{ON},$$

and therefore $$\frac{RP}{OQ}=\frac{PN}{ON}=\tan B.$$

Hence $$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan RPN\tan B}=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

(2) Taking the figure of Art. 45, we have

$$\tan(A-B)=\frac{MP}{OM}=\frac{QN-PR}{OQ+NR}$$

$$=\frac{\frac{QN}{OQ}-\frac{PR}{OQ}}{1+\frac{NR}{OQ}}=\frac{\tan A-\frac{PR}{OQ}}{1+\frac{NR}{PR}\frac{PR}{OQ}}.$$

But, since the angles RPN and NOQ are equal, we have

$$\frac{RP}{PN}=\frac{OQ}{ON},$$

and therefore $$\frac{PR}{OQ}=\frac{PN}{ON}=\tan B.$$

Hence $$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan RPN\tan B}=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

54. As particular cases of the preceding formulae, we have, by putting B equal to $45°$,

$$\tan(A+45°)=\frac{\tan A+1}{1-\tan A}=\frac{1+\tan A}{1-\tan A},$$

and $$\tan(A-45°)=\frac{\tan A-1}{1+\tan A}.$$

Similarly, as in Art. 52, we may prove that

$$\cot(A+B)=\frac{\cot A\cot B-1}{\cot A+\cot B}$$

and $$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}.$$

55. Ex. 1. $\tan 75°=\tan(45°+30°)=\dfrac{\tan 45°+\tan 30°}{1-\tan 45°\tan 30°}$

$$=\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}=\frac{(\sqrt{3}+1)^2}{3-1}=\frac{4+2\sqrt{3}}{2}=2+\sqrt{3}$$

$$=2+1\cdot73205\ldots=3\cdot73205\ldots.$$

Ex. 2. $\tan 15° = \tan(45° - 30°) = \dfrac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$

$$= \frac{1 - \dfrac{1}{\sqrt{3}}}{1 + \dfrac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{(\sqrt{3} - 1)^2}{3 - 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

$$= 2 - 1{\cdot}73205\ldots = {\cdot}26795\ldots.$$

EXAMPLES. XI.

1. If $\tan A = 3$, $\tan B = 2$, find the values of $\tan(A+B)$ and $\tan(A-B)$.

2. If $\tan A = \dfrac{24}{7}$, $\tan B = \dfrac{3}{4}$, find the values of $\tan(A+B)$ and $\tan(A-B)$.

3. If $\sin A = \dfrac{4}{5}$ and $\sin B = \dfrac{8}{17}$, find the value of $\tan(A+B)$.

4. If $\tan \alpha = \dfrac{5}{6}$ and $\tan \beta = \dfrac{1}{11}$, prove that $\alpha + \beta = 45°$. Verify by a graph and accurate measurement.

5. If $\tan A = \dfrac{1}{2}$ and $\tan B = \dfrac{1}{3}$, find the values of $\tan 2A$, $\tan 2B$, $\tan(2A+B)$ and $\tan(2A-B)$.

6. Construct the acute angles whose tangents are $\dfrac{1}{3}$ and $\dfrac{1}{2}$, and verify by measurement that their sum is 45°.

7. The tangents of two acute angles are respectively 3 and 2; show by a graph, and by calculation, that the tangent of their difference is $\dfrac{1}{7}$.

8. The sine of one acute angle is ·6 and the cosine of another is ·5. Show graphically, and also by calculation, that the sine of their difference is ·39 nearly.

9. Draw the positive angle whose cosine is ·4 and show, both by measurement and calculation, that the sine and cosine of an angle which exceeds it by 45° are ·93 and –·365 nearly.

10. Draw the acute angle whose tangent is 7 and the acute angle whose sine is ·7; and show, both by measurement and calculation, that the sine of their difference is approximately ·61.

11. If $\tan A = \frac{\sqrt{3}}{4 - \sqrt{3}}$ and $\tan B = \frac{\sqrt{3}}{4 + \sqrt{3}}$, prove that

$$\tan (A - B) = \cdot 375.$$

12. If $\tan A = \frac{n}{n+1}$ and $\tan B = \frac{1}{2n+1}$, find $\tan (A + B)$.

Prove that

13. $\tan (45^\circ + \theta) \times \tan (135^\circ + \theta) = -1$.

14. $\cot (45^\circ + \theta) \cot (45^\circ - \theta) = 1$.

56. The formulae of Arts. 44 and 52 can be used to obtain the trigonometrical ratios of the sum of more than two angles.

For example

$$\sin (A + B + C) = \sin (A + B) \cos C + \cos (A + B) \sin C$$
$$= [\sin A \cos B + \cos A \sin B] \cos C$$
$$+ [\cos A \cos B - \sin A \sin B] \times \sin C$$
$$= \sin A \cos B \cos C + \cos A \sin B \cos C$$
$$+ \cos A \cos B \sin C - \sin A \sin B \sin C.$$

So

$$\cos (A + B + C) = \cos (A + B) \cos C - \sin (A + B) \sin C$$
$$= (\cos A \cos B - \sin A \sin B) \cos C$$
$$- (\sin A \cos B + \cos A \sin B) \sin C$$
$$= \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C$$
$$- \sin A \sin B \cos C.$$

Also $$\tan (A + B + C) = \frac{\tan (A + B) + \tan C}{1 - \tan (A + B) \tan C}$$

$$= \frac{\dfrac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \dfrac{\tan A + \tan B}{1 - \tan A \tan B} \tan C}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

CHAPTER VII.

THE TRIGONOMETRICAL RATIOS OF MULTIPLE AND SUBMULTIPLE ANGLES.

57. *To find the trigonometrical ratios of an angle $2A$ in terms of those of the angle A.*

If in the formulae of Art. 44 we put $B=A$, we have

$$\mathbf{sin\,2A} = \sin A\cos A + \cos A\sin A = \mathbf{2\sin A\cos A},$$

$$\mathbf{cos\,2A} = \cos A\cos A - \sin A\sin A = \mathbf{cos^2 A - sin^2 A}$$

$$= (1-\sin^2 A) - \sin^2 A = \mathbf{1 - 2\sin^2 A},$$

and also

$$= \cos^2 A - (1-\cos^2 A) = \mathbf{2\cos^2 A - 1};$$

and

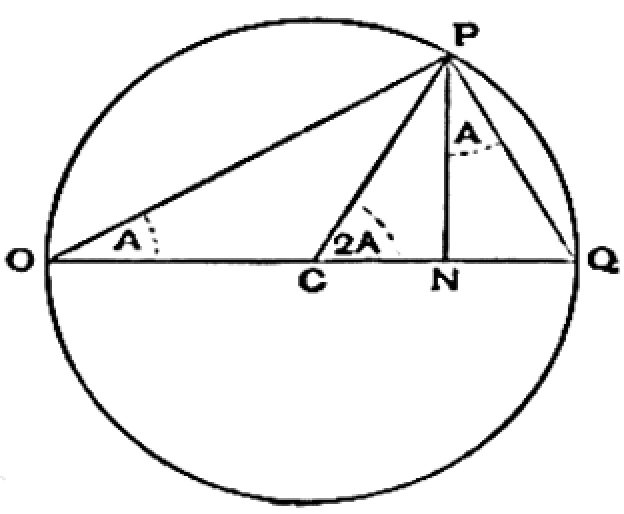
$$\mathbf{tan\,2A} = \frac{\tan A + \tan A}{1 - \tan A\,.\,\tan A} = \frac{\mathbf{2\tan A}}{\mathbf{1-\tan^2 A}}.$$

****58.** An independent geometrical proof of the formulae of the preceding article may be given for values of A which are less than a right angle.

Let QCP be the angle $2A$.

With centre C and radius CP describe a circle, and let QC meet it again in O.

Join OP and PQ, and draw PN perpendicular to OQ.



By geometry, the angle

$$QOP = \frac{1}{2}\angle QCP = A,$$

and the angle

$$NPQ = \angle QOP = A.$$

Hence

$$\sin 2A = \frac{NP}{CP} = \frac{2NP}{2CQ} = 2\frac{NP}{OQ} = 2\frac{NP}{OP}\cdot\frac{OP}{OQ}$$

$= 2 \sin NOP \cos POQ$, since OPQ is a right angle,

$= 2 \sin A \cos A$;

also
$$\cos 2A = \frac{CN}{CP} = \frac{2CN}{OQ} = \frac{(OC+CN)-(OC-CN)}{OQ}$$
$$= \frac{ON-NQ}{OQ} = \frac{ON}{OP}\frac{OP}{OQ} - \frac{NQ}{PQ}\frac{PQ}{OQ}$$
$$= \cos^2 A - \sin^2 A;$$

and
$$\tan 2A = \frac{NP}{CN} = \frac{2NP}{ON-NQ} = \frac{2\frac{NP}{ON}}{1-\frac{NQ}{PN}\frac{PN}{ON}}$$
$$= \frac{2\tan A}{1-\tan^2 A}.$$

59. *To find the trigonometrical functions of 3A in terms of those of A.*

By Art. 44, putting B equal to $2A$, we have

$$\sin 3A = \sin(A+2A) = \sin A \cos 2A + \cos A \sin 2A$$
$$= \sin A\,(1 - 2\sin^2 A) + \cos A \,.\, 2\sin A \cos A,$$

by Art. 57,

$$= \sin A\,(1-2\sin^2 A) + 2\sin A\,(1-\sin^2 A).$$

Hence $\quad \mathbf{\sin 3A = 3 \sin A - 4 \sin^3 A}$ ....(1).

So

$$\cos 3A = \cos(A+2A) = \cos A \cos 2A - \sin A \sin 2A$$
$$= \cos A\,(2\cos^2 A - 1) - \sin A \,.\, 2\sin A \cos A$$
$$= \cos A\,(2\cos^2 A - 1) - 2\cos A\,(1-\cos^2 A).$$

Hence $\quad \mathbf{\cos 3A = 4 \cos^3 A - 3 \cos A}$ ....(2).

Also
$$\tan 3A = \tan(A+2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$= \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \tan A \,.\, \dfrac{2\tan A}{1-\tan^2 A}} = \frac{\tan A\,(1-\tan^2 A) + 2\tan A}{(1-\tan^2 A) - 2\tan^2 A}$$

Hence

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

[The student may find it difficult to remember, and distinguish between, the formulae (1) and (2), which bear a general resemblance to one another, but have their signs in a different order. If in doubt, he may always verify his formula by testing it for a particular case, *e.g.* by putting $A=30°$ for formula (1), and by putting $A=0°$ for formula (2).]

60. By a process similar to that of the last article, the trigonometrical ratios of any higher multiples of θ may be expressed in terms of those of θ. The method is however long and tedious.

As an example, let us express $\cos 5\theta$ in terms of $\cos\theta$. We have

$$\begin{aligned}\cos 5\theta &= \cos(3\theta+2\theta)\\ &= \cos 3\theta\cos 2\theta - \sin 3\theta \sin 2\theta\\ &= (4\cos^3\theta - 3\cos\theta)(2\cos^2\theta - 1) - (3\sin\theta - 4\sin^3\theta) . 2\sin\theta\cos\theta\\ &= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - 2\cos\theta . \sin^2\theta(3 - 4\sin^2\theta)\\ &= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - 2\cos\theta(1-\cos^2\theta)(4\cos^2\theta - 1)\\ &= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - 2\cos\theta(5\cos^2\theta - 4\cos^4\theta - 1)\\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.\end{aligned}$$

EXAMPLES. XII.

1. Find the value of $\sin 2\alpha$ when

(1) $\cos\alpha=\frac{3}{5}$, (2) $\sin\alpha=\frac{12}{13}$, and (3) $\tan\alpha=\frac{16}{63}$.

2. Find the value of $\cos 2\alpha$ when

(1) $\cos\alpha=\frac{15}{17}$, (2) $\sin\alpha=\frac{4}{5}$, and (3) $\tan\alpha=\frac{5}{12}$.

Verify by a graph and accurate measurement.

3. If $\tan\theta=\frac{b}{a}$, find the value of $a\cos 2\theta + b\sin 2\theta$.

Prove that

4. $\frac{\sin 2A}{1+\cos 2A}=\tan A$.

5. $\frac{\sin 2A}{1-\cos 2A}=\cot A$.

6. $\frac{1-\cos 2A}{1+\cos 2A}=\tan^2 A$.

7. $\tan A + \cot A = 2\operatorname{cosec} 2A$.

8. $\tan A - \cot A = -2\cot 2A.$ 9. $\operatorname{cosec} 2A + \cot 2A = \cot A.$

10. $\dfrac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)}=\tan\dfrac{A}{2}\cot\dfrac{B}{2}.$

11. $\dfrac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta}=4\cos 2\theta\cos 4\theta.$

12. $\dfrac{\cos A+\sin A}{\cos A-\sin A}-\dfrac{\cos A-\sin A}{\cos A+\sin A}=2\tan 2A.$

13. $\dfrac{\sin(A+3B)+\sin(3A+B)}{\sin 2A+\sin 2B}=2\cos(A+B).$

14. $\sin 3A+\sin 2A-\sin A=4\sin A\cos\dfrac{A}{2}\cos\dfrac{3A}{2}.$

15. $\dfrac{\sin(n+1)A-\sin(n-1)A}{\cos(n+1)A+2\cos nA+\cos(n-1)A}=\tan\dfrac{A}{2}.$

16. $\dfrac{\sin(n+1)A+2\sin nA+\sin(n-1)A}{\cos(n-1)A-\cos(n+1)A}=\cot\dfrac{A}{2}.$

17. $\dfrac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}=\tan(A+B).$

18. $\dfrac{1}{2}\left(\cot\dfrac{A}{2}-\tan\dfrac{A}{2}\right)=\cot A.$

19. $\dfrac{\cos A}{1-\sin A}=\tan\left(45^\circ+\dfrac{A}{2}\right).$

20. $\tan(45^\circ+\theta)-\tan(45^\circ-\theta)=2\tan 2\theta.$

21. $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}=\tan\theta.$ 22. $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}=\tan\dfrac{\theta}{2}.$

23. $(\sec 2A+1)\sqrt{\sec^2 A-1}=\tan 2A.$

24. $\sin A+\sin B+\sin(A+B)=4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{A+B}{2}.$

25. $\sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)+\sin 2(\alpha-\beta)$
$= -4\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta).$

26. $\cos 4\alpha=1-8\cos^2\alpha+8\cos^4\alpha.$

27. $\sin 4A=4\sin A\cos^3 A-4\cos A\sin^3 A.$

28. $\cos 6\alpha=32\cos^6\alpha-48\cos^4\alpha+18\cos^2\alpha-1.$

29. $\sin\alpha\sin(60^\circ-\alpha)\sin(60^\circ+\alpha)=\dfrac{1}{4}\sin 3\alpha.$

30. $\cos\alpha\cos(60^\circ-\alpha)\cos(60^\circ+\alpha)=\dfrac{1}{4}\cos 3\alpha.$

31. Find, in their simplest forms, the roots of the equation

$$x^2 - 2x \cot 2\beta - 1 = 0.$$

32. Find, in their simplest forms, the roots of the equation

$$x^2 - \sqrt{2} \sin (45^\circ + a) \cdot x + \frac{1}{2} \sin 2a = 0.$$

Submultiple angles.

61. Since the relations of Art. 57 are true for *all* values of the angle A, they will be true if instead of A we substitute $\frac{A}{2}$, and therefore if instead of $2A$ we put $2 \cdot \frac{A}{2}$, *i.e.* A.

Hence we have the relations

$$\mathbf{\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}} \quad \text{.................}(1),$$

$$\mathbf{\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}}$$

$$\mathbf{= 2 \cos^2 \frac{A}{2} - 1 = 1 - 2 \sin^2 \frac{A}{2}} \quad \text{.........}(2),$$

and
$$\mathbf{\tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}} \quad \text{.......................}(3).$$

From (1), we also have

$$\sin A = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2}} = \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}},$$

by dividing numerator and denominator by $\cos^2 \frac{A}{2}$.

So
$$\cos A = \frac{\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2}} = \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$$

62. *To express the trigonometrical ratios of the angle $\frac{A}{2}$ in terms of $\cos A$.*

From equation (2) of the last article, we have

$$\cos A = 1 - 2\sin^2\frac{A}{2},$$

so that $$2\sin^2\frac{A}{2} = 1 - \cos A,$$

and therefore

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}} \quad \text{............(1).}$$

Again, $$\cos A = 2\cos^2\frac{A}{2} - 1,$$

so that $$2\cos^2\frac{A}{2} = 1 + \cos A,$$

and therefore

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}} \quad \text{............(2).}$$

Hence, $$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} \quad \text{.........(3).}$$

63. In each of the preceding formulae it will be noted that there is an ambiguous sign. In any particular case the proper sign can be determined as the following examples will shew.

Ex. 1. *Given* $\cos 45° = \frac{1}{\sqrt{2}}$, *find the values of* $\sin 22\frac{1}{2}°$ *and* $\cos 22\frac{1}{2}°$.

The equation (1) of the last article gives, by putting A equal to 45°,

$$\sin 22\tfrac{1}{2}° = \pm\sqrt{\frac{1-\cos 45°}{2}} = \pm\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \pm\sqrt{\frac{2-\sqrt{2}}{4}}$$
$$= \pm\frac{1}{2}\sqrt{2-\sqrt{2}}.$$

Now $\sin 22\frac{1}{2}°$ is necessarily positive, so that the upper sign must be taken.

Hence $$\sin 22\tfrac{1}{2}^\circ = \frac{1}{2}\sqrt{2-\sqrt{2}}.$$

So $$\cos 22\tfrac{1}{2}^\circ = \pm\sqrt{\frac{1+\cos 45^\circ}{2}} = \pm\sqrt{\frac{2+\sqrt{2}}{4}} = \pm\frac{1}{2}\sqrt{2+\sqrt{2}};$$

also $\cos 22\frac{1}{2}^\circ$ is positive;

$$\therefore \cos 22\tfrac{1}{2}^\circ = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

64. *To express the trigonometrical ratios of the angle $\frac{A}{2}$ in terms of $\sin A$.*

From equation (1) of Art. 60, we have

$$2\sin\frac{A}{2}\cos\frac{A}{2} = \sin A \quad \ldots\ldots\ldots\ldots(1).$$

Also $$\sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1, \text{ always} \quad \ldots\ldots\ldots\ldots(2).$$

First adding these equations, and then subtracting (1) from (2), we have

$$\sin^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2} + \cos^2\frac{A}{2} = 1 + \sin A,$$

and $$\sin^2\frac{A}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2} + \cos^2\frac{A}{2} = 1 - \sin A;$$

i.e. $$\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right)^2 = 1 + \sin A,$$

and $$\left(\sin\frac{A}{2} - \cos\frac{A}{2}\right)^2 = 1 - \sin A;$$

so that $$\sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1+\sin A} \quad \ldots\ldots\ldots\ldots(3),$$

and $$\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{1-\sin A} \quad \ldots\ldots\ldots\ldots(4).$$

By adding, and then subtracting, we have

$$\mathbf{2\sin\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A}} \quad \ldots(5),$$

and $$2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \mp \sqrt{1-\sin A} \quad \ldots(6).$$

The other ratios of $\frac{A}{2}$ are then easily obtained.

65. In each of the formulae (5) and (6) there are two ambiguous signs. In the following example it is shewn how to determine the ambiguity in any particular case.

Ex. *Given that $\sin 30°$ is $\frac{1}{2}$, find the values of $\sin 15°$ and $\cos 15°$.*

Putting $A = 30°$, we have from relations (3) and (4),

$$\sin 15° + \cos 15° = \pm\sqrt{1+\sin 30°} = \pm\frac{\sqrt{3}}{\sqrt{2}},$$

and $$\sin 15° - \cos 15° = \pm\sqrt{1-\sin 30°} = \pm\frac{1}{\sqrt{2}}.$$

Now $\sin 15°$ and $\cos 15°$ are both positive, and $\cos 15°$ is greater than $\sin 15°$. Hence the expressions $\sin 15° + \cos 15°$ and $\sin 15° - \cos 15°$ are respectively positive and negative.

Hence the above two relations should be

$$\sin 15° + \cos 15° = +\frac{\sqrt{3}}{\sqrt{2}},$$

and $$\sin 15° - \cos 15° = -\frac{1}{\sqrt{2}}.$$

Hence $$\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}, \text{ and } \cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

66. *To express the trigonometrical ratios of $\frac{A}{2}$ in terms of $\tan A$.*

From equation (3) of Art. 60, we have

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}.$$

$$\therefore 1-\tan^2\frac{A}{2} = \frac{2}{\tan A}\tan\frac{A}{2}.$$

Hence $$\tan^3\frac{A}{2}+\frac{2}{\tan A}\tan\frac{A}{2}+\frac{1}{\tan^2 A}=1+\frac{1}{\tan^2 A}$$
$$=\frac{1+\tan^2 A}{\tan^2 A}.$$

$$\therefore\ \tan\frac{A}{2}+\frac{1}{\tan A}=\pm\frac{\sqrt{1+\tan^2 A}}{\tan A}.$$

$$\therefore\ \tan\frac{A}{2}=\frac{\pm\sqrt{1+\tan^2 A}-1}{\tan A}\quad\ldots\ldots\ldots(1).$$

67. The ambiguous sign in equation (1) can only be determined when we know something of the magnitude of A.

Ex. *Given* $\tan 15°=2-\sqrt{3}$, *find* $\tan 7\tfrac{1}{2}°$.

Putting $A=15°$ we have, from equation (1) of the last article,

$$\tan 7\tfrac{1}{2}°=\frac{\pm\sqrt{1+(2-\sqrt{3})^2}-1}{2-\sqrt{3}}=\frac{\pm\sqrt{8-4\sqrt{3}}-1}{2-\sqrt{3}}\quad\ldots\ldots\ldots(1).$$

Now $\tan 7\tfrac{1}{2}°$ is positive, so that we must take the upper sign.

Hence $$\tan 7\tfrac{1}{2}°=\frac{+(\sqrt{6}-\sqrt{2})-1}{2-\sqrt{3}}$$

$$=(\sqrt{6}-\sqrt{2}-1)(2+\sqrt{3})=\sqrt{6}-\sqrt{3}+\sqrt{2}-2=(\sqrt{3}-\sqrt{2})(\sqrt{2}-1).$$

EXAMPLES. XIII.

1. If $\sin\theta=\frac{1}{2}$, $\sin\phi=\frac{1}{3}$, and θ and ϕ are acute angles, find the values of $\sin(\theta+\phi)$ and $\sin(2\theta+2\phi)$.

2. The tangent of an acute angle is 2·4. Find its cosecant, the cosecant of half the angle, and the cosecant of the supplement of double the angle.

3. If $\cos\alpha=\frac{3}{5}$ and $\cos\beta=\frac{4}{5}$, find the value of $\cos\frac{\alpha-\beta}{2}$, the angles α and β being positive acute angles. [First find the value of $\cos(\alpha-\beta)$.]

4. If $\cos\alpha=\frac{11}{61}$ and $\sin\beta=\frac{4}{5}$, find the values of $\sin^2\frac{\alpha-\beta}{2}$ and $\cos^2\frac{\alpha+\beta}{2}$, the angles α and β being positive acute angles.

5. Given $\sec\theta = 1\frac{1}{4}$, find $\tan\theta$ and $\tan\frac{\theta}{2}$. Verify by a graph.

6. If $\cos A = \cdot 28$, find the value of $\tan\frac{A}{2}$.

7. Find the values of $\tan 22\frac{1}{2}^\circ$ and $\tan 11\frac{1}{4}^\circ$.

8. If $m\cos A + n\sin A = m$, shew that $\cos\frac{A}{2} = \frac{m}{\sqrt{m^2+n^2}}$.

9. If $\sin\theta + \sin\phi = a$ and $\cos\theta + \cos\phi = b$, find the values of $\cos(\theta - \phi)$ and $\tan\frac{\theta - \phi}{2}$.

Prove that

10. $(\cos\alpha + \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\cos^2\frac{\alpha+\beta}{2}$.

11. $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\frac{\alpha-\beta}{2}$.

12. $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\frac{\alpha-\beta}{2}$.

13. $\sec(45^\circ + \theta)\sec(45^\circ - \theta) = 2\sec 2\theta$.

14. $\tan\left(45^\circ + \frac{A}{2}\right) = \sqrt{\frac{1+\sin A}{1-\sin A}} = \sec A + \tan A$.

15. $\sin^2\left(22\frac{1}{2}^\circ + \frac{A}{2}\right) - \sin^2\left(22\frac{1}{2}^\circ - \frac{A}{2}\right) = \frac{1}{\sqrt{2}}\sin A$.

16. $\frac{1+\tan^2(45^\circ - A)}{1-\tan^2(45^\circ - A)} = \operatorname{cosec} 2A$.

17. $\cos^2\alpha + \cos^2(\alpha + 60^\circ) + \cos^2(\alpha - 60^\circ) = \frac{3}{2}$.

68. By the use of the formulae of the present chapter we can now find the trigonometrical ratios of some important angles.

To find the trigonometrical functions of an angle of 18°.

Let θ stand for 18°, so that 2θ is 36° and 3θ is 54°.

Hence $$2\theta = 90^\circ - 3\theta,$$

and therefore $$\sin 2\theta = \sin(90^\circ - 3\theta) = \cos 3\theta.$$

$$\therefore\ 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta \text{ (Arts. 57 and 59).}$$

Hence, either $\cos\theta = 0$, which gives $\theta = 90^\circ$, or

$$2\sin\theta = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta.$$

$$\therefore\ 4\sin^2\theta + 2\sin\theta = 1.$$

By solving this quadratic equation, we have

$$\sin\theta = \frac{\pm\sqrt{5}-1}{4}.$$

In our case $\sin\theta$ is necessarily a positive quantity. Hence we take the upper sign, and have

$$\mathbf{\sin 18° = \frac{\sqrt{5}-1}{4}}.$$

Hence

$$\cos 18° = \sqrt{1-\sin^2 18°} = \sqrt{1 - \frac{6-2\sqrt{5}}{16}} = \sqrt{\frac{10+2\sqrt{5}}{16}}$$
$$= \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

The remaining trigonometrical ratios of 18° may be now found.

Since 72° is the complement of 18°, the values of the ratios for 72° may be obtained by the use of Art. 24.

69. *To find the trigonometrical functions of an angle of* 36°.

Since $\cos 2\theta = 1 - 2\sin^2\theta$, (Art. 57),

$$\therefore \cos 36° = 1 - 2\sin^2 18° = 1 - 2\left(\frac{6-2\sqrt{5}}{16}\right)$$
$$= 1 - \frac{3-\sqrt{5}}{4},$$

so that

$$\mathbf{\cos 36° = \frac{\sqrt{5}+1}{4}}.$$

Hence

$$\sin 36° = \sqrt{1-\cos^2 36°} = \sqrt{1 - \frac{6+2\sqrt{5}}{16}} = \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

The remaining trigonometrical functions of 36° may now be found.

Also, since 54° is the complement of 36°, the values of the functions for 54° may be found by the help of Art. 24.

EXAMPLES. XIV.

Prove that

1. $\sin^2 72° - \sin^2 60° = \frac{\sqrt{5}-1}{8}$.

2. $\cos^2 48° - \sin^2 12° = \frac{\sqrt{5}+1}{8}$.

3. $\cos 12° + \cos 60° + \cos 84° = \cos 24° + \cos 48°$. Verify by a graph.

4. $\sin 36^\circ \sin 72^\circ \sin 108^\circ \sin 144^\circ = \dfrac{5}{16}$.

5. Two parallel chords of a circle, which are on the same side of the centre, subtend angles of 72° and 144° respectively at the centre. Prove that the perpendicular distance between the chords is half the radius of the circle.

6. In any circle prove that the chord which subtends 108° at the centre is equal to the sum of the two chords which subtend angles of 36° and 60°.

If $A+B+C=2S$, prove that

7. $\sin(S-A)\sin(S-B)+\sin S \sin(S-C)=\sin A \sin B$.

8. $\sin(S-A)+\sin(S-B)+\sin(S-C)-\sin S$
$$=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

9. $\cos^2 S+\cos^2(S-A)+\cos^2(S-B)+\cos^2(S-C)$
$$=2+2\cos A\cos B\cos C.$$

Prove that

10. $\sin(A+B+C+D)+\sin(A+B-C-D)+\sin(A+B-C+D)$
$$+\sin(A+B+C-D)=4\sin(A+B)\cos C\cos D.$$

11. $\sin(A-B)\cos(A+B)+\sin(B-C)\cos(B+C)$
$$+\sin(C-D)\cos(C+D)+\sin(D-A)\cos(D+A)=0.$$

12. $\sin^2\alpha+\sin^2(\alpha-\beta)+\cos\beta\cos(2\alpha-\beta)=1$.

13. $1-\cos^2 x-\cos^2 y+2\cos x\cos y\cos(x+y)=\cos^2(x+y)$.

14. $\cos^2(x+\alpha)+\cos^2(x+\beta)-2\cos(\alpha-\beta)\cos(x+\alpha)\cos(x+\beta)$
$$=\sin^2(\alpha-\beta).$$

CHAPTER VIII.

LOGARITHMS.

70. SUPPOSING that we know that

$$10^{2 \cdot 40312} = 253, \quad 10^{2 \cdot 60959} = 407,$$

and

$$10^{5 \cdot 01271} = 102971,$$

we can shew that $253 \times 407 = 102971$ without performing the operation of multiplication. For

$$\begin{aligned} 253 \times 407 &= 10^{2 \cdot 40312} \times 10^{2 \cdot 60959} \\ &= 10^{2 \cdot 40312 + 2 \cdot 60959} \\ &= 10^{5 \cdot 01271} = 102971. \end{aligned}$$

Here it will be noticed that the process of multiplication has been replaced by the simpler process of addition.

Again, supposing that we know that

$$10^{4 \cdot 90041} = 79507,$$

and that

$$10^{1 \cdot 63347} = 43,$$

we can easily shew that the cube root of 79507 is 43.

For

$$\begin{aligned} \sqrt[3]{79507} &= [79507]^{\frac{1}{3}} = (10^{4 \cdot 90041})^{\frac{1}{3}} \\ &= 10^{\frac{1}{3} \times 4 \cdot 90041} = 10^{1 \cdot 63347} = 43. \end{aligned}$$

Here it will be noticed that the difficult process of extracting the cube root has been replaced by the simpler process of division.

71. Logarithm. Def. *If a be any number, and x and N two other numbers such that $a^x = N$, then x is called the logarithm of N to the base a and is written $\log_a N$.*

The logarithm of a number to a given base is therefore the index of the power to which the base must be raised that it may be equal to the given number.

Exs. Since $10^2 = 100$, therefore $2 = \log_{10} 100$.

Since $10^5 = 100000$, therefore $5 = \log_{10} 100000$.

Since $2^4 = 16$, therefore $4 = \log_2 16$.

Since $8^{\frac{2}{3}} = [8^{\frac{1}{3}}]^2 = 2^2 = 4$, therefore $\frac{2}{3} = \log_8 4$.

Since $9^{-\frac{3}{2}} = \frac{1}{9^{\frac{3}{2}}} = \frac{1}{3^3} = \frac{1}{27}$, therefore $-\frac{3}{2} = \log_9 \left(\frac{1}{27}\right)$.

N.B. Since $a^0 = 1$ always, the logarithm of unity to any base is always zero.

72. In Algebra, if m and n be any real quantities whatever, the following laws, known as the laws of indices, are found to be true:

(i) $a^m \times a^n = a^{m+n}$,

(ii) $a^m \div a^n = a^{m-n}$,

and (iii) $(a^m)^n = a^{mn}$.

Corresponding to these we have three fundamental laws of logarithms, viz.

(i) $\mathbf{\log_a (mn) = \log_a m + \log_a n}$,

(ii) $\mathbf{\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n}$,

and (iii) $\mathbf{\log_a m^n = n \log_a m}$.

The proofs of these laws are given in the following articles.

73. *The logarithm of the product of two quantities is equal to the sum of the logarithms of the quantities to the same base, i.e.*

$$\mathbf{\log_a (mn) = \log_a m + \log_a n}.$$

Let $x = \log_a m$, so that $a^x = m$,

and $y = \log_a n$, so that $a^y = n$.

Then $mn = a^x \times a^y = a^{x+y}$.

$$\therefore \log_a mn = x + y \quad \text{(Art. 71, Def.)}$$
$$= \log_a m + \log_a n.$$

74. *The logarithm of the quotient of two quantities is equal to the difference of their logarithms, i.e.*

$$\mathbf{\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n.}$$

Let $x = \log_a m$, so that $a^x = m$. (Art. 71, Def.)

and $y = \log_a n$, so that $a^y = n$.

Then $\frac{m}{n} = a^x \div a^y = a^{x-y}$.

$$\therefore \log_a \left(\frac{m}{n}\right) = x - y \quad \text{(Art. 71, Def.)}$$
$$= \log_a m - \log_a n.$$

75. *The logarithm of a quantity raised to any power is equal to the logarithm of the quantity multiplied by the index of the power, i.e.*

$$\mathbf{\log_a (m^n) = n \log_a m.}$$

Let $x = \log_a m$, so that $a^x = m$.

Then $m^n = (a^x)^n = a^{nx}$.

$$\therefore \log_a (m^n) = nx \quad \text{(Art. 71, Def.)}$$
$$= n \log_a m.$$

Exs. $\log 48 = \log (2^4 \times 3) = \log 2^4 + \log 3 = 4 \log 2 + \log 3$;

$$\log \frac{63}{484} = \log \frac{7 \times 3^2}{2^2 \times 11^2} = \log 7 + \log 3^2 - \log 2^2 - \log 11^2$$
$$= \log 7 + 2 \log 3 - 2 \log 2 - 2 \log 11;$$

$$\log \sqrt[5]{13} = \log 13^{\frac{1}{5}} = \frac{1}{5} \log 13.$$

76. Common system of logarithms. In the system of logarithms which we practically use the base is always 10, so that, if no base be expressed, the base 10 is always understood. The advantage of using 10 as the base is seen in the three following articles.

77. Characteristic and Mantissa. Def. If the logarithm of any number be partly integral and partly fractional, the integral portion of the logarithm is called its characteristic and the decimal portion is called its mantissa.

Thus, supposing that $\log 795 = 2{\cdot}90037$, the number 2 is the characteristic and $\cdot 90037$ is the mantissa.

Negative characteristics. Suppose we know that

$$\log 2 = \cdot 30103.$$

Then, by Art. 74,

$$\log \tfrac{1}{2} = \log 1 - \log 2 = 0 - \log 2 = - \cdot 30103,$$

so that $\log \frac{1}{2}$ is negative.

Now it is found convenient, as will be seen in Art. 79 that the mantissæ of all logarithms should be kept positive. We therefore instead of $- \cdot 30103$ write $-[1 - \cdot 69897]$, so that

$$\log \tfrac{1}{2} = -(1 - \cdot 69897) = -1 + \cdot 69897.$$

For shortness this latter expression is written $\bar{1}{\cdot}69897$.

The horizontal line over the 1 denotes that the integral part is negative; the decimal part however is positive.

As another example, $\bar{3}{\cdot}47712$ stands for

$$-3 + \cdot 47712.$$

78. *The characteristic of the logarithm of any number can always be determined by inspection.*

(i) Let the number be greater than unity.

Since $10^0 = 1$, therefore $\log 1 = 0$; [Art. 71, Def.]

since $10^1 = 10$, therefore $\log 10 = 1$;

since $10^2 = 100$, therefore $\log 100 = 2$,

and so on.

Hence the logarithm of any number lying between 1 and 10 must lie between 0 and 1, that is, it will be a decimal fraction and therefore have 0 as its characteristic.

So the logarithm of any number between 10 and 100 must lie between 1 and 2, *i.e.* it will have its characteristic equal to 1.

Similarly, the logarithm of any number between 100 and 1000 must lie between 2 and 3, *i.e.* it will have its characteristic equal to 2.

So, if the number lie between 1000 and 10000, the characteristic will be 3.

Generally, *the characteristic of the logarithm of any number will be one less than the number of digits in its integral part.*

Exs. The number 296·3457 has 3 figures in its integral part, and therefore the characteristic of its logarithm is 2.

The characteristic of the logarithm of 29634·57 will be $5-1$, *i.e.* 4.

(ii) Let the number be less than unity.

Since $10^0 = 1$, therefore $\log 1 = 0$;

since $10^{-1} = \frac{1}{10} = \cdot 1$, therefore $\log \cdot 1 = -1$;

since $10^{-2} = \frac{1}{10^2} = \cdot 01$, therefore $\log \cdot 01 = -2$;

since $10^{-3} = \frac{1}{10^3} = \cdot 001$, therefore $\log \cdot 001 = -3$;

and so on.

The logarithm of any number between 1 and ·1 therefore lies between 0 and -1, and so is equal to $-1 +$ some decimal, *i.e.* its characteristic is $\bar{1}$.

So the logarithm of any number between ·1 and ·01 lies between -1 and -2, and hence it is equal to $-2 +$ some decimal, *i.e.* its characteristic is $\bar{2}$.

Similarly, the logarithm of any number between ·01 and ·001 lies between -2 and -3, *i.e.* its characteristic is $\bar{3}$.

Generally, *the characteristic of the logarithm of any decimal fraction will be negative and* **numerically** *will be greater by unity than the number of cyphers following the decimal point.*

For any fraction between 1 and ·1 (*e.g.* ·5) has no cypher following the decimal point and we have seen that its characteristic is $\bar{1}$.

Any fraction between ·1 and ·01 (*e.g.* ·07) has one cypher following the decimal point and we have seen that its characteristic is $\bar{2}$.

Any fraction between ·01 and ·001 (*e.g.* ·003) has two cyphers following the decimal point and we have seen that its characteristic is $\bar{3}$.

Similarly for any fraction.

Exs. The characteristic of the logarithm of the number ·00835 is $\bar{3}$.

The characteristic of the logarithm of the number ·0000053 is $\bar{6}$.

The characteristic of the logarithm of the number ·34567 is $\bar{1}$.

79. *The mantissæ of the logarithm of all numbers, consisting of the same digits, are the same.*

This will be made clear by an example.

Suppose we are given that

$$\log 6682 = 3{\cdot}82491.$$

Then

$$\log 668{\cdot}2 = \log \frac{6682}{10} = \log 6682 - \log 10 \text{ (Art. 74)}$$

$$= 3{\cdot}82491 - 1 = 2{\cdot}82491;$$

$$\log {\cdot}6682 = \log \frac{6682}{10000} = \log 6682 - \log 10000$$

$$\text{(Art. 74)}$$

$$= 3{\cdot}82491 - 4 = \bar{1}{\cdot}82491.$$

So
$$\log {\cdot}0006682 = \log \frac{6682}{10^7} = \log 6682 - \log 10^7$$

$$= 3{\cdot}82491 - 7 = \bar{4}{\cdot}82491.$$

Now the numbers 6682, 668·2, ·6682, and ·0006682 consist of the same significant figures, and only differ in the position of the decimal point. We observe that their logarithms have the same decimal portion, *i.e.* the same mantissa, and they only differ in the characteristic.

The value of this characteristic is in each case determined by the rule of the previous article.

It will be noted that the mantissa of a logarithm is always positive.

80. Tables of logarithms. The logarithms of all numbers from 1 to 108000 are given, correct to seven places of decimals, in Chambers' Tables of Logarithms. Many calculations however need not be made to the extent of accuracy represented by seven places of decimals. Sufficient accuracy for most practical purposes is given by logarithms to five places of decimals—usually called five-figure logarithms. It would be advisable that the student should have access to such a table. At the end of this book will be found a table of four-figure logarithms, which is the largest table that can be conveniently and legibly printed in a book whose pages are of the present size.

81. It will be noted that only the decimal part (*i.e.* the mantissæ) of the logarithms are there given. The proper characteristic must be supplied by the rule of Art. 78.

To obtain the logarithm of any number such as 5264 we proceed thus. Run the eye down the extreme left-hand column of Table I. until it arrives at the number 52. Then look horizontally until the eye sees the figures 7210 which are vertically below the number 6 at the top of the page. We thus get

No.	Log.
526	7210.

The fourth figure in our number is 4. Look at the extreme right of the page of logarithms for the Difference Column. Run the eye down the column headed by 4 until

we arrive at the line which commenced with 52 and there we find the number 3. We thus have

	No.	Log.
	526	7210
Difference for	4	3
	5264	7213.

Now, by Art. 78, the characteristic of the logarithm of 5264 is 3.

$$\therefore \log 5264 = 3{\cdot}7213.$$

82. If we had a number of five figures, we for the extreme right-hand figure take the corresponding number in the Difference Column and place it one place to the right of the column. Thus suppose we wanted log 52647. As before

	No.	Log.
	5264	7213
Diff. for	7 =	6
	52647	72136,

so that $\log 52647 = 4{\cdot}72136.$

The extreme right-hand figure is however unreliable when we use four-figure tables. The latter cannot be relied upon further than the fourth place of decimals.

83. As another example, we shall take from Table I. the logarithm of 9876. We find

	No.	Log.
	987	9943
Difference for	6	3

$$\therefore \log 9876 = 3{\cdot}9946.$$

If we used five-figure tables instead, we should have

	No.	Log.
	987	99432
Difference for	6	26

$$\therefore \log 9876 = 3{\cdot}99458.$$

It will thus be noted that, in general, whatever be the table we use the logarithm given corresponding to any number is, strictly speaking, only an approximation; but an approximation which is more and more near according to the number of decimal places in the table we use. Thus a four-figure table is correct to four places; a seven-figure table to seven places, etc.

84. We shall now work a few numerical examples to show the efficiency of the application of logarithms for purposes of calculation.

Ex. 1. *Find the value of* $\sqrt[5]{23{\cdot}4}$.

Let $x=\sqrt[5]{23{\cdot}4}=(23{\cdot}4)^{\frac{1}{5}}$,

so that $\log x=\frac{1}{5}\log(23{\cdot}4)$, by Art. 75.

In a five-figure table of logarithms we find, opposite the number 234, the logarithm 36922.

Hence $\log 23{\cdot}4=1{\cdot}36922.$

Therefore $\log x=\frac{1}{5}[1{\cdot}36922]={\cdot}27384.$

Again, in the table of logarithms we find, corresponding to the logarithm 27384, the number 18786, so that

$$\log 1{\cdot}8786={\cdot}27384.$$

$$\therefore\ x=1{\cdot}8786.$$

Ex. 2. *Using four-figure logarithms* (*Table I.*) *find the value of*

$$\frac{(6{\cdot}45)^3\times\sqrt[3]{{\cdot}00034}}{(9{\cdot}37)^2\times\sqrt[4]{8{\cdot}93}}.$$

Let x be the required value so that, by Arts. 74 and 75,

$$\log x=\log(6{\cdot}45)^3+\log({\cdot}00034)^{\frac{1}{3}}-\log(9{\cdot}37)^2-\log\sqrt[4]{8{\cdot}93}$$

$$=3\log(6{\cdot}45)+\frac{1}{3}\log({\cdot}00034)-2\log(9{\cdot}37)-\frac{1}{4}\log 8{\cdot}93.$$

Now in Table I. we find

opposite the number 645 the logarithm 8096,
" " " 34 " " 5315,
" " " 937 " " 9717,
and " " " 893 " " 9509.

Hence $$\log x = 3 \times \cdot 8096 + \frac{1}{3}(\bar{4}\cdot 5315)$$
$$- 2 \times \cdot 9717 - \frac{1}{4} \times \cdot 9509.$$

But $$\frac{1}{3}(\bar{4}\cdot 5315) = \frac{1}{3}[\bar{6} + 2\cdot 5315]$$
$$= \bar{2} + \cdot 8438.$$

$$\begin{aligned} \therefore \log x &= 2\cdot 4288 + [\bar{2} + \cdot 8438] - 1\cdot 9434 - \cdot 2377 \\ &= 3\cdot 2726 - 4\cdot 1811 \\ &= \bar{1} + 4\cdot 2726 - 4\cdot 1811 \\ &= \bar{1}\cdot 0915. \end{aligned}$$

But, from Table I.,

$$\begin{array}{lrr} & \log 123 & = 2\cdot 0899 \\ \text{diff. for} & 5 = & 17 \\ \hline & \therefore \log 1235 & = 3\cdot 0916 \end{array}$$

$$\therefore x = \cdot 1235 \text{ nearly.}$$

When the logarithm of any number does not quite agree with any logarithm in the tables, but lies between two consecutive logarithms, it will be shewn in the next chapter how the number may be accurately found.

Ex. 3. *Having given* $\log 2 = \cdot 30103$, *find the number of digits in* 2^{67} *and the position of the first significant figure in* 2^{-37}.

We have $$\log 2^{67} = 67 \times \log 2 = 67 \times \cdot 30103$$
$$= 20\cdot 16901.$$

Since the characteristic of the logarithm of 2^{67} is 20, it follows, by Art. 78, that in 2^{67} there are 21 digits.

Again, $$\log 2^{-37} = -37 \log 2 = -37 \times \cdot 30103$$
$$= -11\cdot 13811 = \overline{12}\cdot 86189.$$

Hence, by Art. 78, in 2^{-37} there are 11 cyphers following the decimal point, *i.e.* the first significant figure is in the twelfth place of decimals.

EXAMPLES. XV.

1. Given $\log 4 = \cdot 60206$ and $\log 3 = \cdot 4771213$, find the logarithms of $\cdot 8$, $\cdot 003$, $\cdot 0108$, and $(\cdot 00018)^{\frac{1}{7}}$.

2. Given $\log 11 = 1\cdot 0413927$ and $\log 13 = 1\cdot 1139434$, find the values of (1) $\log 1\cdot 43$, (2) $\log 133\cdot 1$, (3) $\log \sqrt[4]{143}$, and (4) $\log \sqrt[3]{\cdot 00169}$.

3. What are the characteristics of the logarithms of 243·7, ·0153, 2·8713, ·00057, ·023, $\sqrt[5]{24615}$, and $(24589)^{\frac{3}{4}}$?

4. Find the 5th root of ·003, having given $\log 3 = \cdot 47712$ and

$$\log 3129 = 3 \cdot 49542.$$

5. Find the value of (1) $7^{\frac{1}{7}}$, (2) $(84)^{\frac{2}{5}}$, and (3) $(\cdot 021)^{\frac{1}{5}}$, having given

$$\log 2 = \cdot 30103, \ \log 3 = \cdot 47712,$$
$$\log 7 = \cdot 84510, \ \log 13205 = 4 \cdot 12073,$$
$$\log 58845 = 4 \cdot 76971 \text{ and } \log 4618 = 3 \cdot 66444.$$

6. Having given $\log 3 = \cdot 47712$,

find the number of digits in

$$(1)\ 3^{43}, \quad (2)\ 3^{27}, \text{ and } (3)\ 3^{62},$$

and the position of the first significant figure in

$$(4)\ 3^{-13}, \quad (5)\ 3^{-43}, \text{ and } (6)\ 3^{-65}.$$

7. From the Tables find the seventh root of ·00002675.

Making use of the Tables, find the approximate values of

8. $\sqrt[3]{645 \cdot 3}$.

9. $\sqrt[5]{8236}$.

10. $\dfrac{\sqrt{5} \times \sqrt[3]{7}}{\sqrt[4]{8} \times \sqrt[5]{9}}$.

11. $\sqrt[3]{\dfrac{7 \cdot 2 \times 8 \cdot 3}{9 \cdot 4 \div 16 \cdot 5}}$.

12. $\sqrt{\dfrac{8^{\frac{1}{5}} \times 11^{\frac{1}{3}}}{\sqrt{74} \times \sqrt[5]{62}}}$.

13. $2\pi\sqrt{\dfrac{l}{g}}$, where $\pi = 3 \cdot 1416$, $l = 37$, and $g = 32 \cdot 2$.

14. pv^{γ}, where $p = 345$, $v = 2 \cdot 62$, and $\gamma = 1 \cdot 4$.

15. the volume of a sphere of radius 43 centimetres, given that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$, where $\pi = 3 \cdot 1416$.

16. Draw the graph of $\log_{10} x$.

CHAPTER IX.

TABLES OF LOGARITHMS AND TRIGONOMETRICAL RATIOS. PRINCIPLE OF PROPORTIONAL PARTS.

85. WE have pointed out that the logarithms of all numbers from 1 to 108000 may be found in Chambers' Mathematical Tables, so that, for example, the logarithms of 74583 and 74584 may be obtained directly therefrom.

Suppose however we wanted the logarithm of a number lying between these two, *e.g.* the number 74583·3.

To obtain the logarithm of this number we use the Principle of Proportional Parts which states that *the increase in the logarithm of a number is proportional to the increase in the number itself.*

Thus in a seven-figure table we find

$$\log 74583 = 4{\cdot}8726398 \quad \ldots\ldots\ldots\ldots\ldots(1),$$

and

$$\log 74584 = 4{\cdot}8726457 \quad \ldots\ldots\ldots\ldots\ldots(2).$$

The quantity $\log 74583{\cdot}3$ will clearly lie between $\log 74583$ and $\log 74584$.

Let then $\qquad \log 74583{\cdot}3 = \log 74583 + x$

$$= 4{\cdot}8726398 + x \ldots\ldots\ldots\ldots(3).$$

From (1) and (2), we see that for an increase 1 in the number the increase in the logarithm is ·0000059.

The Theory of Proportional Parts then states that for

an increase of ·3 in the number the increase in the logarithm is

$$\cdot 3 \times \cdot 0000059, \textit{ i.e., } \cdot 00000177.$$

$$\text{Hence} \quad \log 74583{\cdot}3 = 4{\cdot}8726398 + {\cdot}00000177$$
$$= 4{\cdot}87264157.$$

86. As another example, we shall find the value of log ·0382757 and shall exhibit the working in a more concise form.

From a seven-figure table we obtain

$$\log \cdot 038275 = \bar{2}{\cdot}5829152$$
$$\log \cdot 038276 = \bar{2}{\cdot}5829265.$$

Hence the difference for

$$\cdot 000001 = \cdot 0000113.$$

Therefore the difference for

$$\cdot 0000007 = \cdot 7 \times \cdot 0000113$$
$$= \cdot 00000791.$$

$$\therefore \log \cdot 0382757 = \bar{2}{\cdot}5829152$$
$$+ \quad \cdot 00000791$$
$$= \bar{2}{\cdot}58292311.$$

Since we only require logarithms to seven places of decimals, we omit the last digit and the answer is $\bar{2}{\cdot}5829231$.

87. The converse question is often met with, viz., to find the number whose logarithm is given. If the logarithm be one of those tabulated the required number is easily found. The method to be followed when this is not the case is shewn in the following examples.

Find the number whose logarithm is 2·6283924.

On reference to the seven-figure tables we find that the logarithm 6283924 is not tabulated, but that the nearest logarithms are 6283889 and 6283991, between which our logarithm lies.

We have then $\log 425{\cdot}00 = 2{\cdot}6283889$ ....(1),

and $\log 425{\cdot}01 = 2{\cdot}6283991$ ....(2).

Let $\log (425{\cdot}00 + x) = 2{\cdot}6283924$ ....(3).

From (1) and (2), we see that corresponding to a difference ·01 in the number there is a difference ·0000102 in the logarithm.

From (1) and (3), we see that corresponding to a difference x in the number there is a difference ·0000035 in the logarithm.

Hence we have $x : {\cdot}01 :: {\cdot}0000035 : 0000102$.

$$\therefore\ x = \frac{35}{102} \times {\cdot}01 = \frac{{\cdot}35}{102} = {\cdot}0034\ldots.$$

Hence the required number $= 425{\cdot}00 + {\cdot}0034 = 425{\cdot}0034$.

88. Where logarithms are taken out of the tables the labour of subtracting successive logarithms may be avoided. As explained in the last chapter there is in all logarithm tables a column headed *Diff.* on the extreme right of each page.

As an example, let us find the logarithm of 527·46.

From Table I., we have

$$\begin{aligned} \log 527 \quad &= 2{\cdot}7218 \\ \text{diff. for } \ {\cdot}4 \quad &= \quad\ \ 3 \\ \text{diff. for } \ {\cdot}06 \quad & \\ \left(= \frac{1}{10} \times \text{diff. for } {\cdot}6\right) &= \quad\ \ 05 \\ \hline \therefore\ \log 527{\cdot}46 &= 2{\cdot}72215 \end{aligned}$$

This is as near as we can get the result from a four-figure table.

We shall solve two more examples, taking all the logarithms from Table I., and only putting down the necessary steps.

Ex. 1. *Find the seventh root of* ·03457.

Let $x = ({\cdot}03457)^{\frac{1}{7}}$.

$$\begin{aligned} \therefore\ \log x &= \frac{1}{7}\log({\cdot}03457) = \frac{1}{7}(\bar{2}{\cdot}5387) \\ &= \frac{1}{7}(\bar{7} + 5{\cdot}5387) \\ &= \bar{1}{\cdot}79124. \end{aligned}$$

But, from Table I.,

$$\log \cdot 618 \quad = \bar{1}\cdot 7910$$

diff. for $\quad 3 = \quad 2$

diff. for $\quad 6 = \quad 4$

$$\therefore \log \cdot 61836 = \bar{1}\cdot 79124.$$

$$\therefore x = \cdot 61836.$$

This result will not be reliable to the fifth place of decimals.

Ex. 2. *If $a = 345\cdot 6$ and $b = 283\cdot 5$, find the value of the square root of $a^2 - b^2$.*

Let
$$x^2 = a^2 - b^2 = (a-b)(a+b).$$

$$\therefore 2 \log x = \log (a-b) + \log (a+b)$$
$$= \log 62\cdot 1 + \log 629\cdot 1.$$

Now, by Table I.,

$$\log 62\cdot 1 \quad = 1\cdot 7931$$
$$\log 629\cdot \quad = 2\cdot 7987$$

diff. for $\quad \cdot 1 = \quad 1$

Therefore, by addition, $\quad 2 \log x = 4\cdot 5919.$

$$\therefore \log x = 2\cdot 29595.$$

But, from Table I., $\quad \log 197 \quad = 2\cdot 2945$

diff. for $\quad \cdot 6 = \quad 13$

diff. for $\quad \cdot 07 = \quad 16$

$$\therefore \log 197\cdot 67 = 2\cdot 29596$$
$$= \log x, \text{ very nearly.}$$

$$\therefore x = 197\cdot 67 \text{ approx.}$$

89. The proof of the Principle of Proportional Parts cannot be given at this stage. It is not strictly true without certain limitations.

If the numbers to which the principle is applied contain not less than five significant figures, then we may rely on the result as correct to seven places of decimals.

For example, we must *not* apply the principle to obtain the value of log 2·5 from the values of log 2 and log 3.

For, if we did, since these logarithms are ·30103 and ·4771213, the logarithm of 2·5 would be ·389075.

But from the tables the value of log $2\cdot5$ is found to be $\cdot3979400$.

Hence the result which we should obtain would be manifestly quite incorrect.

So, if the numbers to which the principle is applied contain not less than four digits, we can rely on the result as correct to five places of decimals.

Similarly, if the numbers contain not less than 3 digits, the result can be relied upon in general to four places of decimals.

EXAMPLES. XVI.

1. Given $\log 35705 = 4\cdot5527290$,
and $\log 35706 = 4\cdot5527412$,
find the values of $\log 35705\cdot7$ and $\log 35\cdot70585$.

2. Given $\log 5\cdot8742 = \cdot7689487$
and $\log 587\cdot43 = 2\cdot7689561$,
find the values of $\log 58742\cdot57$ and $\log \cdot00587422$.

3. Given $\log 47847 = 4\cdot6798547$
and $\log 47848 = 4\cdot6798638$,
find the numbers whose logarithms are respectively $2\cdot6798593$ and $\bar{3}\cdot6798617$.

4. Given $\log 258\cdot36 = 2\cdot4122253$
and $\log 2\cdot5837 = \cdot4122421$,
find the numbers whose logarithms are $\cdot4122378$ and $\bar{2}\cdot4122287$.

Using Tables, find the value of

5. $(3\cdot4578)^{\frac{1}{5}}$. **6.** $(2\cdot3894)^{\frac{1}{7}}$.

7. $(\cdot6349)^{\frac{1}{5}} \times (3\cdot825)^3$. **8.** $(4\cdot2357)^2 \div (2\cdot3896)^3$.

9. $(789\cdot42)^2 \div (53\cdot47)^2$.

10. $\sqrt[3]{\dfrac{(4\cdot3758)^2 \times 3\cdot87}{(4\cdot382)^4}}$. **11.** $\dfrac{(8\cdot97)^3 \times (\cdot0235)^{\frac{2}{5}}}{(3\cdot4894)^{\frac{2}{3}} \div 6\cdot829}$.

90. In Chambers' Tables will be found tables giving the values of the trigonometrical ratios of angles between $0°$ and $45°$, the angles increasing by differences of $1'$.

It is unnecessary to separately tabulate the ratios for angles between $45°$ and $90°$, since the ratios of angles between $45°$ and $90°$ can be reduced to those of angles between $0°$ and $45°$. (Art. 39.)

For example,

$$[\sin 76° \, 11' = \sin (90° - 13° \, 49') = \cos 13° \, 49',$$

and is therefore known].

Such a table is called a table of natural sines, cosines, etc. to distinguish it from the table of logarithmic sines, cosines, etc.

If we want to find the sine of an angle which contains an integral number of degrees and minutes, we can obtain it from the tables. If, however, the angle contain seconds, we must use the principle of proportional parts.

Ex. 1. *Given* $\sin 29° \, 14' = \cdot 48837$,

and $\sin 29° \, 15' = \cdot 48862$,

find the value of $\sin 29° \, 14' \, 32''$.

By subtraction we have

$$\text{difference in the sine for } 1' = \cdot 00025.$$

$$\therefore \text{ difference in the sine for } 32'' = \frac{32}{60} \times \cdot 00025 = \cdot 00013,$$

$$\begin{aligned} \therefore \sin 29° \, 14' \, 32' &= \cdot 48837 \\ &\quad + \cdot 00013 \\ &= \cdot 48850. \end{aligned}$$

Ex. 2. *Given* $\cos 16° \, 27' = \cdot 95907$,

and $\cos 16° \, 28' = \cdot 95899$,

find $\cos 16° \, 27' \, 47''$.

We note that the cosine decreases as the angle increases.

Hence for an **increase** of $1'$, *i.e.* $60''$, in the angle, there is a **decrease** of $\cdot 00008$ in the cosine.

Hence for an **increase** of 47″ in the angle, there is a **decrease** of $\frac{47}{60} \times \cdot 00008$ in the cosine.

$$\therefore \cos 16^\circ 27' 47'' = \cdot 95907 - \frac{47}{60} \times \cdot 00008$$

$$= \cdot 95907 - \cdot 00006$$

$$= \cdot 95907$$
$$- \cdot 00006$$

$$= \cdot 95901.$$

In practice this may be abbreviated thus;

$$\cos 16^\circ 28' = \quad \cdot 95899$$
$$\cos 16^\circ 27' = \quad \cdot 95907$$

$$\text{diff. for } 1' = - \cdot 00008.$$

$$\therefore \quad \text{diff. for } 47'' = - \frac{47}{60} \times \cdot 00008$$

$$= - \cdot 00006.$$

$$\therefore \quad \text{Ans.} = \quad \cdot 95907$$
$$- \quad \cdot 00006$$

$$= \quad \cdot 95901.$$

91. The inverse question, to find the angle, when one of its trigonometrical ratios is given, will now be easy.

Ex. *Find the angle whose cotangent is* 1·41093, *having given* $\cot 35^\circ 19' = 1 \cdot 41148$, *and* $\cot 35^\circ 20' = 1 \cdot 41061$.

Let the required angle be $35^\circ 19' + x''$,

so that $\cot (35^\circ 19' + x'') = 1 \cdot 41093.$

From these three equations we have

For an increase of 60″ in the angle, a decrease of ·00087 in the cotangent,

For an increase of x'' in the angle, a decrease of ·00055 in the cotangent.

$$\therefore x : 60 :: 55 : 87, \text{ so that } x = 38 \text{ nearly.}$$

Hence the required angle $= 35^\circ 19' 38''$.

92. In working all questions involving the application of the Principle of Proportional Parts, the student must be very careful to note whether the trigonometrical ratios increase or decrease as the angle increases. As a help to his memory, he may observe that in the first quadrant the three trigonometrical ratios whose names begin with co-, *i.e.* the cosine, the cotangent, and the cosecant, all decrease as the angle increases.

Tables of logarithmic sines, cosines, etc.

93. In many kinds of trigonometric calculation, as in the solution of triangles, we often require the logarithms of trigonometrical ratios. To avoid the inconvenience of first finding the sine of any angle from the tables and then obtaining the logarithm of this sine by a second application of the tables, it has been found desirable to have separate tables giving the logarithms of the various trigonometrical functions of angles. As before, it is only necessary to construct the tables for angles between 0° and 45°.

Since the sine of an angle is always less than unity, the logarithm of its sine is always negative (Art. 78).

Again, since the tangent of an angle between 0° and 45° is less than unity its logarithm is negative, whilst the logarithm of the tangent of an angle between 45° and 90° is the logarithm of a number greater than unity and is therefore positive.

94. To avoid the trouble and inconvenience of printing the proper sign to the logarithms of the trigonometric functions, the logarithms as tabulated are not the true logarithms, but the true logarithms *increased by* 10.

For example, sine $30^\circ = \frac{1}{2}$.

Hence
$$\log \sin 30^\circ = \log \tfrac{1}{2} = -\log 2$$
$$= -\cdot 30103 = \bar{1}\cdot 69897.$$

The logarithm tabulated is therefore

$$10 + \log \sin 30^\circ, \quad i.e.\ 9\cdot 69897.$$

Again $\tan 60^\circ = \sqrt{3}$.

Hence $\log \tan 60^\circ = \frac{1}{2} \log 3 = \frac{1}{2}(\cdot 4771213)$

$= \cdot 2385606$.

The seven-figure logarithm tabulated is therefore

$$10 + \cdot 2385606, \text{ i.e. } 10\cdot 2385606.$$

The symbol L is used to denote these "tabular logarithms," *i.e.* the logarithms as found in the English books of tables.

Thus $L \sin 15^\circ\, 25' = 10 + \log \sin 15^\circ\, 25'$,

and $L \sec 48^\circ\, 23' = 10 + \log \sec 48^\circ\, 23'$.

95. If we want to find the tabular logarithm of any function of an angle, which contains an integral number of degrees and minutes, we can obtain it directly from the tables. If, however, the angle contains seconds we must use the principle of proportional parts. The method of procedure is similar to that of Art. 90. We give an example and also one of the inverse question.

Ex. 1. *Given* $L\, cosec\, 32^\circ\, 21' = 10\cdot 27157$,

and $L\, cosec\, 32^\circ\, 22' = 10\cdot 27137$,

find $L\, cosec\, 32^\circ\, 21'\, 51''$.

For an increase of $60''$ in the angle, there is a *decrease* of $\cdot 00020$ in the logarithm.

Hence for an increase of $51''$ in the angle, the corresponding *decrease* is $\frac{51}{60} \times \cdot 00020$, *i.e.* $\cdot 00017$.

Hence $L \operatorname{cosec} 32^\circ\, 51'\, 51'' = 10\cdot 27157$

$- \quad \cdot 00017$

$= 10\cdot 27140$.

Ex. 2. *Using five-figure logarithms, find the angle such that the tabular logarithm of its tangent is* $9\cdot 44172$.

From the tables, we have

$$\begin{array}{rl} L \tan x = & 9\cdot 44172 \\ L \tan 15^\circ 27' = & 9\cdot 44151 \\ \hline \text{diff.} = & 21. \end{array} \qquad \begin{array}{rl} L \tan 15^\circ 30' = & 9\cdot 44299 \\ L \tan 15^\circ 27' = & 9\cdot 44151 \\ \hline \text{diff. for } 3' = & 148. \end{array}$$

$$\therefore \text{ corresponding increase} = \frac{21}{148} \times 3'$$

$$= \frac{21}{148} \times 180''$$

$$= 26'' \text{ nearly.}$$

$$\therefore \; x = 15^\circ 27' 26''.$$

Ex. 3. *Given* $\qquad L \sin 14^\circ 6' = 9{\cdot}3867,$
find L cosec $14^\circ 6'$.

Here $\qquad \log \sin 14^\circ 6' = L \sin 14^\circ 6' - 10$

$$= -1 + {\cdot}3867.$$

Now $\qquad \log \operatorname{cosec} 14^\circ 6' = \log \frac{1}{\sin 14^\circ 6'}$

$$= -\log \sin 14^\circ 6'$$

$$= 1 - {\cdot}3867 = {\cdot}6133.$$

Hence $\qquad L \operatorname{cosec} 14^\circ 6' = 10{\cdot}6133.$

More generally, we have $\sin\theta \times \operatorname{cosec}\theta = 1$.

$$\therefore \; \log \sin\theta + \log \operatorname{cosec}\theta = 0.$$

$$\therefore \; L \sin\theta + L \operatorname{cosec}\theta = 20.$$

The error to be avoided is this; the student sometimes assumes that, because

$$\log \operatorname{cosec} 14^\circ 6' = -\log \sin 14^\circ 6',$$

he may therefore assume that

$$L \operatorname{cosec} 14^\circ 6' = -L \sin 14^\circ 6'.$$

This is obviously untrue.

EXAMPLES. XVII.

1. Given $\qquad \sin 43^\circ 23' = {\cdot}68688$
and $\qquad \sin 43^\circ 24' = {\cdot}68709,$
find the value of $\qquad \sin 43^\circ 23' 47''.$

2. Find also the angle whose sine is ·68703.

3. Given $\qquad \sin 18^\circ 15' = {\cdot}31316$
and $\qquad \sin 18^\circ 16' = {\cdot}31344,$
find $\sin 18^\circ 15' 37''$ and the angle whose sine is ·31329.

4. Given $\cos 32°16' = \cdot 84557$

and $\cos 32°17' = \cdot 84542$,

find the values of $\cos 32°16'24''$ and of $\cos 32°16'47''$.

5. Find also the angles whose cosines are

$$\cdot 84548 \text{ and } \cdot 84552.$$

6. Given $\tan 76°21' = 4\cdot 11778$

and $\tan 76°22' = 4\cdot 12301$,

find the values of $\tan 76°21'29''$ and $\tan 76°21'47''$.

7. Given $\operatorname{cosec} 13°8' = 4\cdot 40106$

and $\operatorname{cosec} 13°9' = 4\cdot 39558$,

find the values of $\operatorname{cosec} 13°8'19''$ and $\operatorname{cosec} 13°8'37''$.

8. Find also the angle whose cosecant is $4\cdot 39679$.

9. Given $L \sin 33°27' = 9\cdot 74132$

and $L \sin 33°28' = 9\cdot 74151$,

find $L \sin 33°27'49''$ and the angle whose $L \sin$ is $9\cdot 74140$.

10. Given $L \cos 34°44' = 9\cdot 9147729$

and $L \cos 34°45' = 9\cdot 9146852$,

find the value of $L \cos 34°44'27''$.

11. Find also the angle θ, where

$$L \cos \theta = 9\cdot 9147328.$$

12. Given $L \cot 71°27' = 9\cdot 5257779$

and $L \cot 71°28' = 9\cdot 5253589$,

find the value of $L \cot 71°27'47''$,

and solve the equation $L \cot \theta = 9\cdot 5254782$.

13. Given $L \sec 18°27' = 10\cdot 0229168$

and $L \sec 18°28' = 10\cdot 0229590$,

find the value of $L \sec 18°27'35''$.

14. Find also the angle whose $L \sec$ is $10\cdot 0229285$.

15. Find in degrees, minutes, and seconds the angle whose sine is $\cdot 6$, given that

$$\log 6 = \cdot 7781513, \quad L \sin 36°52' = 9\cdot 7781186,$$

and $$L \sin 36°53' = 9\cdot 7782870.$$

96. Tables II. and III. at the end of this book give the sines and tangents of all angles between 0° and 90°. These will give the trigonometrical ratios of all angles.

Ex. *Find from these tables the value of sin* $21° \, 37' \, 25''$.

Run the eye down the extreme left-hand column till we arrive at 21°, and then horizontally till we get to the number 3665 which is vertically below 30 in the top line.

Thus $\sin 21° \, 30' = \cdot 3665.$

Then, in the difference column, we have the number 19 corresponding to the heading 7′.

$$\begin{array}{lr} \text{Thus} \qquad \sin 21° \, 30' = & \cdot 3665 \\ \text{diff. for} \qquad 7' = & 19 \\ \text{diff. for } 25'' = \dfrac{25}{60} \times \text{diff. for } 1' = \dfrac{25}{60} \times 3 = & 1 \cdot 25 \\ \hline \therefore \sin 21° \, 37' \, 25'' = & \cdot 3685. \end{array}$$

This is as near an answer as we can get with these tables.

Ex. *Find similarly tan* $70° \, 26' \, 43''$.

$$\begin{array}{lr} \tan 70° \, 20' = & 2 \cdot 798 \\ \text{diff. for} \qquad 6' = & 16 \\ \text{diff. for } 43'' = \dfrac{43}{60} \times \text{diff. for } 1' & \\ = \dfrac{43}{60} \times 3 = & 2 \cdot 15 \\ \hline & 2 \cdot 816 \end{array}$$

Running our eye down the extreme left-hand column of Table III. until we get to 70, and then horizontally we find under 20′ the number 2·798.

In the same line in the column headed "Differences" we find 16 corresponding to 6′.

Also by proportional parts the diff. for $43'' = \frac{43}{60} \times$ that for 1′, and the diff. for 1′ from the same line is 3.

On working out this proportion, we get 2·15.

Hence the work stands as above, and 2·816 is the nearest approximation we can get from this table.

97. Similarly Tables IV. and V. give the logarithms of the sines and tangents of all angles from 0° to 90°; we can thus obtain the logarithms of all the trigonometrical functions.

Ex. 1. *From these tables find* $L \cos 25° \, 37' \, 32''$.

By Art. 39, the cosine of any angle = the sine of its complement.

$$\therefore L \cos 25° \, 37' \, 32'' = L \sin (90° - 25° \, 37' \, 32'')$$
$$= L \sin (64° \, 22' \, 28'').$$

Using Table IV., similarly as in Art. 96, we have

$$L \sin 64° \, 20' = 9{\cdot}9549$$

$$\text{Diff. for} \quad 2' = \quad 1$$

$$\text{Diff. for } 28'' = \frac{28}{60} \times \text{Diff. for } 1'$$

$$= \frac{28}{60} \times 1 \qquad = \qquad \Big| \; 47$$

$$9{\cdot}9550 \; \Big| \; 47$$

$$\therefore L \sin 64° \, 22' \, 38'' = 9{\cdot}9550,$$

as nearly as we can get from these tables.

Ex. 2. *Find* x *where* $L \cot x = 10{\cdot}1461$.

From Table V.,

$$L \tan 54° \, 20' = 10{\cdot}1441.$$

$$\text{Diff. for} \quad 7' = \quad 19$$

$$L \tan 54° \, 27' = 10{\cdot}1460$$

The difference between this and the given number above $= 1$ and the diff. for $1'$ is, from Table V., 3.

$$\therefore \text{ diff. in } \angle = \frac{1}{3} \times 1' = 20''.$$

$$\therefore L \tan 54^\circ\, 27'\, 20'' = 10{\cdot}1461.$$

$$\therefore L \cot x = L \tan 54^\circ\, 27'\, 20''$$

$$= L \cot 35^\circ\, 32'\, 40'' \qquad \text{(Art. 39).}$$

$$\therefore x = 35^\circ\, 32'\, 40'',$$

as nearly as we can get from four-figure tables.

Ex. 3. *Find x, where $\sec x = 1{\cdot}824$.*

If we have tables of secants then x may be found as before.

But we may also use Tables II. or IV.

For $L \cos x = 10 + \log \dfrac{1}{1{\cdot}824}$

$= 10 - \log 1{\cdot}824 = 10 - {\cdot}2610$, by Table I.
$= 9{\cdot}7390.$

By Table IV.,

$$L \sin 33^\circ\, 10' = 9{\cdot}7380$$

$$\text{Diff. for } \quad 5' = \quad 10$$

$$\therefore L \sin 33^\circ\, 15' = 9{\cdot}7390$$

$$\therefore L \cos x = L \sin 33^\circ\, 15'$$

$$= L \cos 56^\circ\, 45' \quad \text{[Art. 39].}$$

$$\therefore x = 56^\circ\, 45'.$$

EXAMPLES. XVIII.

[For the following examples tables will be required; either the four-figure tables at the end of this book, or other tables with a higher number of places of decimals. If four-figure tables are used, the result may be correct only to the nearest tenth of a minute.]

Find the value of

1. $\sin 37^\circ 46'$. **2.** $\tan 73^\circ 29'$.

3. $\cos 65°37'$.

4. $\sin 28°17'20''$.

5. $\tan 59°44'25''$.

6. $\cos 72°29'37''$.

7. $L\sin 15°27'20''$.

8. $L\cos 37°16'$.

9. $L\tan 25°17'15''$.

Find the value of θ to the nearest minute, where

10. $\cos\theta = \cdot 9725$.

11. $\sin\theta = \frac{3}{8}$.

12. $\operatorname{cosec}\theta = 1\cdot 0998$.

13. $L\tan\theta = 9\cdot 6197$.

14. $L\cos\theta = 9\cdot 993$.

15. $L\sec\theta = 10\cdot 15$.

Find θ when

16. $\sin\theta = \cdot 38472$.

17. $\cos\theta = \cdot 48937$.

18. $\sec\theta = 5\cdot 3824$.

19. $L\tan\theta = 10\cdot 52896$.

20. $L\sin\theta = 9\cdot 38526$.

21. $L\cos\theta = 9\cdot 62835$.

22. $L\cot\theta = 10\cdot 84352$.

Find that value of θ, between $0°$ and $90°$, which satisfies the equation, to the nearest minute,

23. $\cos\theta = \sin 29°15' \times \tan 52°30'$.

24. $\sin\theta = \sqrt{\sin 66°50'}$.

25. Draw the graph of $\log_{10} x$ for the values $x = 370, 371, 372$ and 373 and hence estimate the values of $\log 370\cdot 6$ and $372\cdot 4$.

CHAPTER X.

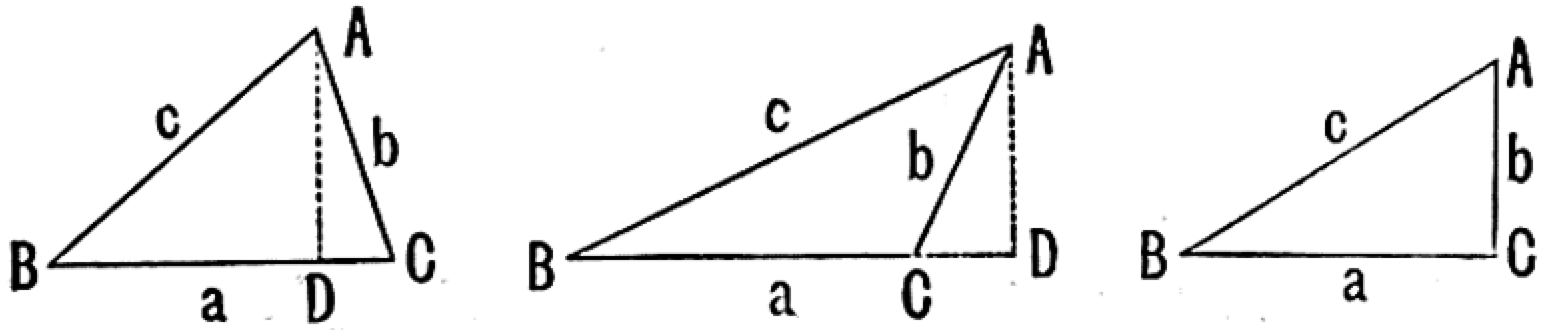
RELATIONS BETWEEN THE SIDES AND THE TRIGONOMETRICAL RATIOS OF THE ANGLES OF ANY TRIANGLE.

98. In any triangle ABC, the side BC, opposite to the angle A, is denoted by a; the sides CA and AB, opposite to the angles B and C respectively, are denoted by b and c.

99. Theorem. *In any triangle ABC,*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

i.e. the sines of the angles are proportional to the opposite sides.



Draw AD perpendicular to the opposite side meeting it, produced if necessary, in the point D.

In the triangle ABD, we have

$$\frac{AD}{AB} = \sin B, \text{ so that } AD = c \sin B.$$

In the triangle ACD, we have

$$\frac{AD}{AC} = \sin C, \text{ so that } AD = b \sin C.$$

[If the angle C be obtuse, as in the second figure, we have

$$\frac{AD}{b} = \sin ACD = \sin (180° - C) = \sin C \qquad \text{(Art. 42)},$$

so that $$AD = b \sin C.]$$

Equating these two values of AD, we have

$$c \sin B = b \sin C,$$

i.e. $$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

In a similar manner, by drawing a perpendicular from B upon CA, we have

$$\frac{\sin C}{c} = \frac{\sin A}{a}.$$

If one of the angles, C, be a right angle, as in the third figure, we have $\sin C = 1$,

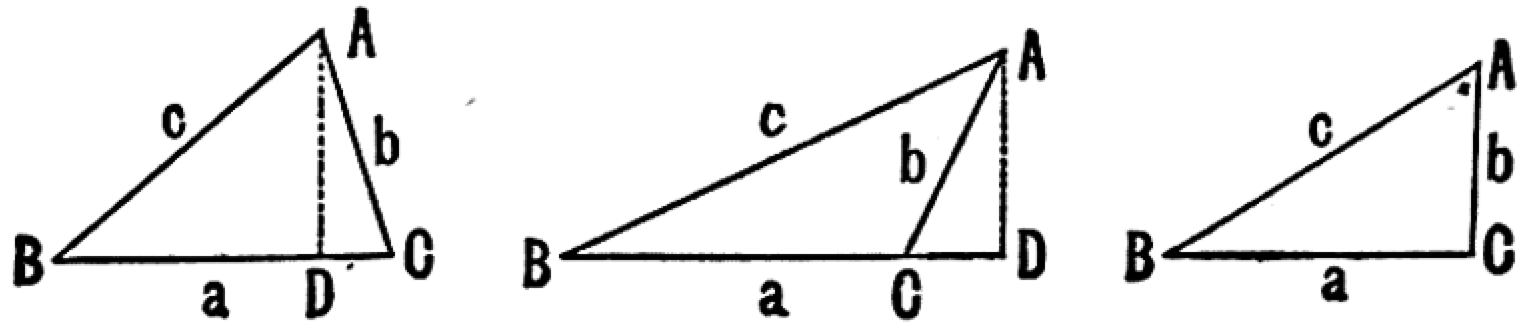
$$\sin A = \frac{a}{c}, \text{ and } \sin B = \frac{b}{c}.$$

Hence $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{1}{c} = \frac{\sin C}{c}.$$

We therefore have, in all cases,

$$\mathbf{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}.$$

100. *In any triangle, to find the cosine of an angle in terms of the sides.*



Let ABC be the triangle and let the perpendicular from A on BC meet it, produced if necessary, in the point D.

First, let the angle C be **acute**, as in the first figure.

By geometry, we have

$$AB^2 = BC^2 + CA^2 - 2BC \, . \, CD \ldots\ldots\ldots\ldots \text{(i)}.$$

But $\dfrac{CD}{CA} = \cos C$, so that $CD = b \cos C$.

Hence (i) becomes

$$c^2 = a^2 + b^2 - 2a \, . \, b \cos C,$$

i.e.
$$2ab \cos C = a^2 + b^2 - c^2,$$

i.e.
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Secondly, let the angle C be **obtuse**, as in the second figure.

By geometry, we have

$$AB^2 = BC^2 + CA^2 + 2BC \, . \, CD \ldots\ldots\ldots\ldots \text{(ii)}.$$

But $\dfrac{CD}{CA} = \cos ACD = \cos (180° - C) = -\cos C,$

(Art. 42)

so that
$$CD = -b \cos C.$$

Hence (ii) becomes

$$c^2 = a^2 + b^2 + 2a(-b \cos C) = a^2 + b^2 - 2ab \cos C,$$

so that, as in the first case, we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

In a similar manner it may be shewn that

$$\mathbf{\cos A = \frac{b^2 + c^2 - a^2}{2bc}}, \quad \text{and} \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}.$$

Also
$$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A},$$

and
$$b^2 = c^2 + a^2 - 2ca \cos B.$$

If one of the angles, C, be a right angle, the above formula would give $c^2 = a^2 + b^2$, so that $\cos C = 0$. This is correct, since C is a right angle.

The above formula is therefore true for all values of C.

Ex. If $a=15$, $b=36$, and $c=39$,
then

$$\cos A = \frac{36^2+39^2-15^2}{2\times 36\times 39} = \frac{3^2(12^2+13^2-5^2)}{2\times 3^2\times 12\times 13} = \frac{288}{24\times 13} = \frac{12}{13} = \cdot 923077\ldots$$

Hence, from the tables, we obtain $A = 22°37'$ nearly.

101. *To find the sines of half the angles in terms of the sides.*

In any triangle we have, by Art. 100,

$$\cos A = \frac{b^2+c^2-a^2}{2bc}.$$

By Art. 61, we have

$$\cos A = 1 - 2\sin^2\frac{A}{2}.$$

Hence $\quad 2\sin^2\frac{A}{2} = 1 - \cos A = 1 - \frac{b^2+c^2-a^2}{2bc}$

$$= \frac{2bc-b^2-c^2+a^2}{2bc} = \frac{a^2-(b^2+c^2-2bc)}{2bc} = \frac{a^2-(b-c)^2}{2bc}$$

$$= \frac{[a+(b-c)][a-(b-c)]}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc} \quad \ldots(1).$$

Let $2s$ stand for $a+b+c$, so that s is equal to half the sum of the sides of the triangle, *i.e.* s is equal to the semi-perimeter of the triangle.

We then have

$$a+b-c = a+b+c-2c = 2s-2c = 2(s-c),$$

and $\quad a-b+c = a+b+c-2b = 2s-2b = 2(s-b).$

The relation (1) therefore becomes

$$2\sin^2\frac{A}{2} = \frac{2(s-c)\times 2(s-b)}{2bc} = 2\,\frac{(s-b)(s-c)}{bc}.$$

$$\therefore\ \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad \ldots\ldots\ldots(2).$$

Similarly,

$$\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}},\text{ and }\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}.$$

102. *To find the cosines of half the angles in terms of the sides.*

By Art. 61, we have

$$\cos A = 2\cos^2\frac{A}{2}-1.$$

Hence $\quad 2\cos^2\frac{A}{2}=1+\cos A=1+\frac{b^2+c^2-a^2}{2bc}$

$$=\frac{2bc+b^2+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{[(b+c)+a][(b+c)-a]}{2bc}=\frac{(a+b+c)(b+c-a)}{2bc}\quad\ldots(1).$$

Now $\quad b+c-a=a+b+c-2a=2s-2a=2(s-a)$,

so that (1) becomes

$$2\cos^2\frac{A}{2}=\frac{2s\times 2(s-a)}{2bc}=2\,\frac{s(s-a)}{bc}.$$

$$\therefore\quad \mathbf{\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}}\quad\ldots\ldots\ldots\ldots(2).$$

Similarly,

$$\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}},\text{ and }\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}.$$

103. *To find the tangents of half the angles in terms of the sides.*

Since $$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}},$$

we have, by (2) of Arts. 101 and 102,

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \div \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Similarly,

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \text{ and } \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Since, in a triangle, A is always $< 180°$, $\frac{A}{2}$ is always $< 90°$.

The sine, cosine, and tangent of $\frac{A}{2}$ are therefore always positive (Art. 37).

The positive sign must therefore always be prefixed to the radical sign in the formulae of this and the last two articles.

104. Ex. If $a=13$, $b=14$, and $c=15$,

then $$s=\frac{13+14+15}{2}=21,\ s-a=8,\ s-b=7,$$

and $$s-c=6.$$

Hence $$\sin\frac{A}{2}=\sqrt{\frac{7\times 6}{14\times 15}}=\frac{1}{\sqrt{5}}=\frac{1}{5}\sqrt{5},$$

$$\sin\frac{B}{2}=\sqrt{\frac{6\times 8}{15\times 13}}=\frac{4}{\sqrt{65}}=\frac{4}{65}\sqrt{65},$$

$$\cos\frac{C}{2}=\sqrt{\frac{21\times 6}{13\times 14}}=\frac{3}{\sqrt{13}}=\frac{3}{13}\sqrt{13},$$

and $$\tan\frac{B}{2}=\sqrt{\frac{6\times 8}{21\times 7}}=\frac{4}{7}.$$

105. *To express the sine of any angle of a triangle in terms of the sides.*

We have, by Art. 61,

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}.$$

But, by the previous articles,

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},\ \text{and}\ \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}.$$

Hence

$$\sin A=2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}.$$

$$\therefore\ \mathbf{\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.}$$

EXAMPLES. XIX.

In a triangle

1. Given $a=25$, $b=52$, and $c=63$,

find $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, and $\tan\frac{C}{2}$.

2. Given $a=125$, $b=123$, and $c=62$,

find the sines of half the angles and the sines of the angles.

3. Given $a=18$, $b=24$, and $c=30$,

find $\sin A$, $\sin B$, and $\sin C$.

Verify by a graph.

4. Given $a=35$, $b=84$, and $c=91$,

find $\tan A$, $\tan B$, and $\tan C$.

5. Given $a=13$, $b=14$, and $c=15$,

find the sines of the angles. Verify by a graph.

6. Given $a=287$, $b=816$, and $c=865$,

find the values of $\tan\frac{A}{2}$ and $\tan A$.

7. Given $a=51$, $b=40$, and $c=13$,

find $\cos A$ and the value of A to the nearest minute.

8. Find, by help of the tables, the size to the nearest minute of the angles of a triangle whose sides are 3, 4, and 6 inches.

106. *In any triangle, to prove that*

$$a = b\cos C + c\cos B.$$

Take the figures of Art. 100.

In the first case, we have

$$\frac{BD}{BA} = \cos B, \text{ so that } BD = c\cos B,$$

and
$$\frac{CD}{CA} = \cos C, \text{ so that } CD = b\cos C.$$

Hence $a = BC = BD + DC = c\cos B + b\cos C.$

In the second case, we have

$$\frac{BD}{BA} = \cos B, \text{ so that } BD = c\cos B,$$

and
$$\frac{CD}{CA} = \cos ACD = \cos(180° - C)$$

$$= -\cos C \quad \text{(Art. 42)},$$

so that
$$CD = -b\cos C.$$

Hence, in this case,

$$a = BC = BD - CD = c\cos B - (-b\cos C),$$

so that in each case

$$\mathbf{a = b\cos C + c\cos B.}$$

Similarly, $b = c\cos A + a\cos C,$

and $c = a\cos B + b\cos A$

107. *In any triangle, to prove that*

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}.$$

In any triangle, we have

$$\frac{b}{c} = \frac{\sin B}{\sin C}.$$

$$\therefore \frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2\cos\dfrac{B+C}{2}\sin\dfrac{B-C}{2}}{2\sin\dfrac{B+C}{2}\cos\dfrac{B-C}{2}} \text{ (Art. 48)}$$

$$= \frac{\tan\dfrac{B-C}{2}}{\tan\dfrac{B+C}{2}} = \frac{\tan\dfrac{B-C}{2}}{\tan\left(90^\circ - \dfrac{A}{2}\right)}$$

$$= \frac{\tan\dfrac{B-C}{2}}{\cot\dfrac{A}{2}} \text{ (Art. 39).}$$

Hence $$\mathbf{\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}}.$$

108. The Student will often meet with identities, which he is required to prove, which involve both the sides and the angles of a triangle.

It is, in general, desirable to substitute in the identity for the sides in terms of the angles, or to substitute for the trigonometrical ratios of the angles in terms of the sides.

Ex. 1. *Prove that* $a\cos\dfrac{B-C}{2} = (b+c)\sin\dfrac{A}{2}$.

By Art. 99, we have

$$\frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A} = \frac{2\sin\dfrac{B+C}{2}\cos\dfrac{B-C}{2}}{2\sin\dfrac{A}{2}\cos\dfrac{A}{2}}.$$

Now $$A+B+C=180^\circ,$$

so that $$\frac{B+C}{2} = 90^\circ - \frac{A}{2},$$

and hence $$\sin\frac{B+C}{2} = \cos\frac{A}{2}. \qquad \text{[Art. 39.]}$$

Hence $$\frac{b+c}{a}=\frac{\cos\frac{A}{2}\cos\frac{B-C}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}.$$

$$\therefore\ (b+c)\sin\frac{A}{2}=a\cos\frac{B-C}{2}.$$

Ex. 2. *In any triangle prove that*

$$(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0.$$

By Art. 99, we have

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k \text{ (say)}.$$

Hence the given expression

$$=(b^2-c^2)\frac{\cos A}{ak}+(c^2-a^2)\frac{\cos B}{bk}+(a^2-b^2)\frac{\cos C}{ck}$$

$$=\frac{1}{k}\left[(b^2-c^2)\frac{b^2+c^2-a^2}{2abc}+(c^2-a^2)\frac{c^2+a^2-b^2}{2abc}+(a^2-b^2)\frac{a^2+b^2-c^2}{2abc}\right],$$

by Art. 100,

$$=\frac{1}{2abck}[b^4-c^4-a^2(b^2-c^2)+c^4-a^4-b^2(c^2-a^2)+a^4-b^4-c^2(a^2-b^2)]$$

$$=0.$$

Ex. 3. *In any triangle prove that*

$$(a+b+c)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=2c\cot\frac{C}{2}.$$

The left-hand member

$$=2s\left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}+\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}\right], \qquad \text{by Art. 103,}$$

$$=2s\sqrt{\frac{s-c}{s}}\left[\sqrt{\frac{s-b}{s-a}}+\sqrt{\frac{s-a}{s-b}}\right]=2\sqrt{s(s-c)}\left[\frac{s-b+s-a}{\sqrt{(s-a)(s-b)}}\right]$$

$$=\frac{2\sqrt{s(s-c)}\,.\,c}{\sqrt{(s-a)(s-b)}}, \text{ since } 2s=a+b+c,$$

$$=2c\cot\frac{C}{2}.$$

EXAMPLES. XX.

In any triangle ABC, prove that

1. $\sin\frac{B-C}{2}=\frac{b-c}{a}\cos\frac{A}{2}$.

2. $\frac{a+b}{a-b}=\tan\frac{A+B}{2}\cot\frac{A-B}{2}$.

3. $a\sin\left(\frac{A}{2}+B\right)=(b+c)\sin\frac{A}{2}$.

4. $b^2\cos^2 C-c^2\cos^2 B=b^2-c^2$.

5. $b^2\sin 2C+c^2\sin 2B=2bc\sin A$.

6. $a(b\cos C-c\cos B)=b^2-c^2$.

7. $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$.

8. $a(\cos B+\cos C)=2(b+c)\sin^2\frac{A}{2}$.

9. $a(\cos C-\cos B)=2(b-c)\cos^2\frac{A}{2}$.

10. $\frac{\sin(B-C)}{\sin(B+C)}=\frac{b^2-c^2}{a^2}$.

11. $\frac{a+b-c}{a+b+c}=\tan\frac{A}{2}\tan\frac{B}{2}$.

12. $(b+c-a)\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)=2a\cot\frac{A}{2}$.

13. $a^2+b^2+c^2=2(bc\cos A+ca\cos B+ab\cos C)$.

14. $(a^2-b^2+c^2)\tan B=(a^2+b^2-c^2)\tan C$.

15. $c^2=(a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2}$.

16. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$.

17. $a\cos A+b\cos B=c\cos(A-B)$.

18. $(a+c)\tan\frac{B}{2}+(a-c)\cot\frac{B}{2}=2c\cot C$.

19. In a triangle whose sides are 3, 4, and $\sqrt{38}$ feet respectively, prove that the largest angle is greater than 120°.

20. The sides of a right-angled triangle are 21 and 28 feet; find the length of the perpendicular drawn to the hypothenuse from the right angle.

21. If in any triangle the angles be to one another as $1:2:3$, prove that the corresponding sides are as $1:\sqrt{3}:2$.

22. The perpendicular AD to the base of a triangle ABC divides it into segments such that BD, CD, and AD are in the ratio of 2, 3, and 6; prove that the vertical angle of the triangle is 45°.

23. In any triangle ABC if D be any point of the base BC such that

$$BD : DC :: m : n,$$

prove that

$$(m+n)\cot ADC = n\cot B - m\cot C,$$

and

$$(m+n)^2 AD^2 = (m+n)(mb^2+nc^2) - mna^2.$$

109. Identities holding between the trigonometrical ratios of the angles of a triangle.

When three angles A, B, and C, are such that their sum is 180°, many identical relations are found to hold between their trigonometrical ratios.

The method of proof is best seen from the following examples.

Ex. 1. *If $A+B+C=180°$, to prove that*

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C.$$

$$\sin 2A + \sin 2B + \sin 2C$$
$$= 2\sin(A+B)\cos(A-B) + 2\sin C\cos C. \quad \text{(Arts. 48, 57)}$$

Since $$A+B+C=180°,$$

we have $$A+B=180°-C,$$

and therefore $$\sin(A+B)=\sin C,$$

and $$\cos(A+B)=-\cos C. \quad \text{(Art. 42)}$$

Hence the expression

$$=2\sin C\cos(A-B)+2\sin C\cos C$$
$$=2\sin C[\cos(A-B)+\cos C]$$
$$=2\sin C[\cos(A-B)-\cos(A+B)]$$
$$=2\sin C \cdot 2\sin A\sin B$$
$$=4\sin A\sin B\sin C.$$

Ex. 2. *If* $$A+B+C=180^\circ,$$

prove that $$\cos A+\cos B-\cos C=-1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

The expression $=\cos A+(\cos B-\cos C)$

$$=2\cos^2\frac{A}{2}-1+2\sin\frac{B+C}{2}\sin\frac{C-B}{2}. \qquad \text{(Arts. 57, 48)}$$

Now $$B+C=180^\circ-A,$$

so that $$\frac{B+C}{2}=90^\circ-\frac{A}{2},$$

and therefore $$\sin\frac{B+C}{2}=\cos\frac{A}{2},$$

and $$\cos\frac{B+C}{2}=\sin\frac{A}{2}.$$

Hence the expression

$$=2\cos^2\frac{A}{2}-1+2\cos\frac{A}{2}\sin\frac{C-B}{2}$$

$$=2\cos\frac{A}{2}\left[\cos\frac{A}{2}+\sin\frac{C-B}{2}\right]-1$$

$$=2\cos\frac{A}{2}\left[\sin\frac{B+C}{2}+\sin\frac{C-B}{2}\right]-1$$

$$=2\cos\frac{A}{2}\,.\,2\sin\frac{C}{2}\cos\frac{B}{2}-1$$

$$=-1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

Ex. 3. *If* $$A+B+C=180^\circ,$$

prove that $\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C.$

Let $$S=\sin^2 A+\sin^2 B+\sin^2 C,$$

so that
$$2S=2\sin^2 A+1-\cos 2B+1-\cos 2C \qquad \text{(Art. 57)}$$
$$=2\sin^2 A+2-2\cos(B+C)\cos(B-C) \qquad \text{(Art. 48)}$$
$$=2-2\cos^2 A+2-2\cos(B+C)\cos(B-C)$$
$$=4+2\cos A\cos(B+C)+2\cos A\cos(B-C),$$

since $$\cos A=\cos\{180^\circ-(B+C)\}=-\cos(B+C).$$

$$\therefore\ S=2+\cos A[\cos(B-C)+\cos(B+C)]$$
$$=2+\cos A\,.\,2\cos B\cos C$$
$$=2+2\cos A\cos B\cos C.$$

Ex. 4. *If* $A+B+C=180°$,

prove that $\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

$$\tan(A+B)=\tan(180°-C)=-\tan C. \qquad \text{(Art. 42)}$$

$$\therefore \frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C.$$

$$\therefore \tan A+\tan B=-\tan C+\tan A\tan B\tan C,$$

i.e. $$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

EXAMPLES. XXI.

If $A+B+C=180°$, prove that

1. $\sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C$.

2. $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$.

3. $\cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C$.

4. $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

5. $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$.

6. $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

7. $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C$.

8. $\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C$.

9. $\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\cos C$.

10. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

11. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$.

12. $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$.

13. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

14. $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$

15. $\sin(B+2C) + \sin(C+2A) + \sin(A+2B)$

$$= 4 \sin \frac{B-C}{2} \sin \frac{C-A}{2} \sin \frac{A-B}{2}.$$

16. $$\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

17. $\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C)$

$$= 4 \sin A \sin B \sin C.$$

18. $\cot A + \cot B + \cot C - \cot A \cot B \cot C = \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C.$

CHAPTER XI.

SOLUTION OF TRIANGLES.

110. In any triangle the three sides and the three angles are often called the elements of the triangle. When any three elements of the triangle are given, provided they be not the three angles, the triangle is in general completely known, *i.e.* its other angles and sides can be calculated. When the three angles are given, only the ratios of the lengths of the sides can be found, so that the triangle is given in *shape* only and not in *size*. When three elements of a triangle are given the process of calculating its other three elements is called the **Solution of the Triangle.**

We shall first discuss the solution of right-angled triangles, *i.e.* triangles which have one angle given equal to a right angle.

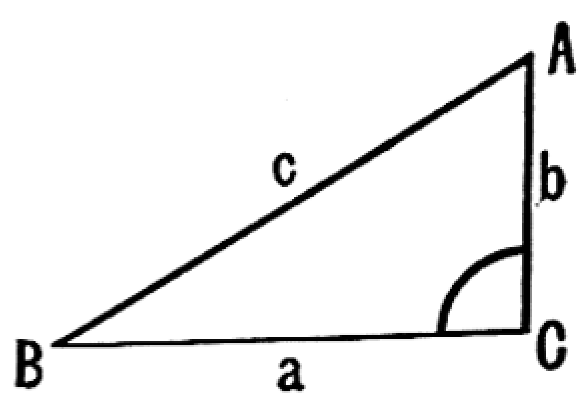
The next four articles refer to such triangles, and C denotes the right angle.

111. *Case I. Given the hypothenuse and one side, to solve the triangle.*

Let b be the given side and c the given hypothenuse.

The angle B is given by the relation

$$\sin B = \frac{b}{c}.$$



Sin B is thus a known decimal and the value of B can be found by

help of Table II. If b and c are large numbers it may be easier to take logarithms, and then we have

$$L\sin B = 10 + \log b - \log c.$$

Since b and c are known, we have $L\sin B$ and therefore B.

The angle A $(=90° - B)$ is then known.

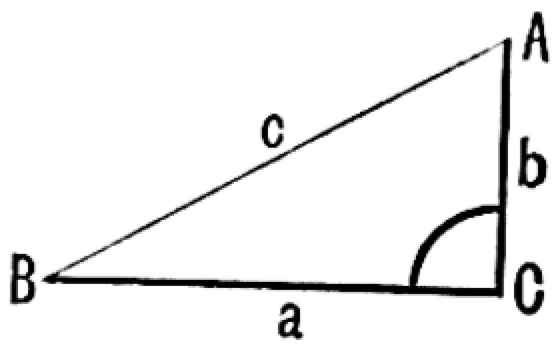
The side a is obtained from either of the relations

$$\cos B = \frac{a}{c}, \quad \tan B = \frac{b}{a}, \quad \text{or} \quad a = \sqrt{(c-b)(c+b)}.$$

112. *Case II. Given the two sides a and b, to solve the triangle.*

Here B is given by

$$\tan B = \frac{b}{a}.$$



Hence B is known by Table III; or if b and c be large numbers we have, by taking logarithms,

$$L\tan B = 10 + \log b - \log a.$$

Hence $L\tan B$, and therefore B, is known.

The angle A $(= 90° - B)$ is then known.

The hypothenuse c is given by the relation $c = \sqrt{a^2 + b^2}$.

This relation is not however very suitable for logarithmic calculation, and c is best given by

$$\sin B = \frac{b}{c}, \quad i.e. \quad c = \frac{b}{\sin B}.$$

$$\therefore \quad \log c = \log b - \log \sin B$$

$$= 10 + \log b - L\sin B.$$

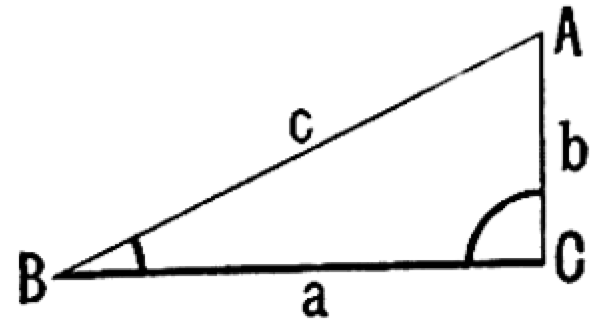
Hence c is obtained.

113. *Case III. Given an angle B and one of the sides a, to solve the triangle.*

Here A $(= 90^\circ - B)$ is known.

The side b is found from the relation

$$\frac{b}{a} = \tan B,$$



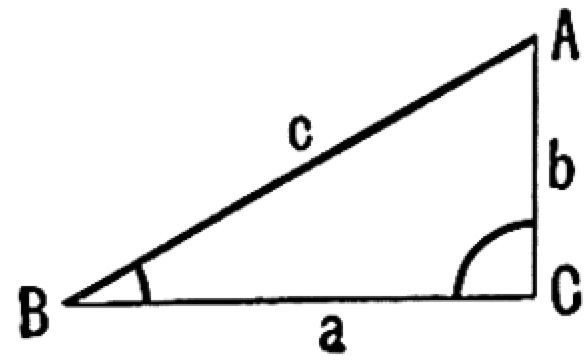
and c from the relation

$$\frac{a}{c} = \cos B.$$

114. *Case IV. Given an angle B and the hypothenuse c, to solve the triangle.*

Here A is known, and a and b are obtained from the relations

$$\frac{a}{c} = \cos B, \text{ and } \frac{b}{c} = \sin B.$$



EXAMPLES. XXII.

1. In a right-angled triangle ABC, where C is the right angle, if $a = 50$ and $B = 75^\circ$, find the sides. $(\tan 75^\circ = 2 + \sqrt{3}.)$

2. Solve the triangle of which two sides are equal to 10 and 20 feet and of which the included angle is 90°; given that

$$\tan 26^\circ 33' = \cdot 49967$$

and

$$\tan 26^\circ 36' = \cdot 50076.$$

3. If $a = 324 \cdot 5$ ft., $b = 658 \cdot 7$ ft., and $C = 90^\circ$, find A and c, by the help of the tables.

4. A road rises vertically one foot for every six feet of its length; find, by the help of the tables, its inclination to the horizon to the nearest minute.

5. A road is inclined to the horizon at a uniform inclination of 10°; find how much it rises vertically in 100 yards of its length.

6. The angular elevation of the sun above the horizon is 37°; find the length of the shadow of a tower whose height is 150 feet.

7. The length of the perpendicular from one angle of a triangle upon the base is 3 inches and the lengths of the sides containing this angle are 4 and 5 inches. Find the angles, having given

$$\log 2 = \cdot 30103,\ \log 3 = \cdot 4771213,$$

$$L \sin 36^\circ 52' = 9\cdot 7781186, \text{ diff. for } 1' = 1684,$$

and $$L \sin 48^\circ 35' = 9\cdot 8750142, \text{ diff. for } 1' = 1115.$$

8. Find the acute angles of a right-angled triangle whose hypothenuse is four times as long as the perpendicular drawn to it from the opposite angle.

115. We now proceed to the case of the triangle which is not given to be right angled.

The different cases to be considered are;

Case I. The three sides given;

Case II. Two sides and the included angle given;

Case III. Two sides and the angle opposite one of them given;

Case IV. One side and two angles given;

Case V. The three angles given.

116. *Case I. The three sides a, b, and c given.*

Since the sides are known, the semi-perimeter s is known and hence also the quantities $s-a$, $s-b$, and $s-c$.

The half-angles $\frac{A}{2}$, $\frac{B}{2}$, and $\frac{C}{2}$ are then found from the formulae

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\quad \tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}},$$

and $$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Only two of the angles need be found, the third being known since the sum of the three angles is always 180°.

The angles may also be found by using the formulae for the sine or cosine of the semi-angles.

(Arts. 101 and 102.)

The above formulae are all suited for logarithmic computation.

The angle A may also be obtained from the formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}. \quad \text{(Art. 100.)}$$

This formula is not, in general, suitable for logarithmic calculation. It may be conveniently used however when the sides a, b, and c are small numbers.

117. **Ex. 1.** *The sides of a triangle are* 7, 10, *and* 13 *feet; using the formula for the cosine of an angle in terms of the sides and the tables at the end of this book, find the greatest angle.*

Here $a = 7$, $b = 10$ and $c = 13$.

$$\therefore \cos C = \frac{7^2 + 10^2 - 13^2}{2 \times 7 \times 10} = \frac{49 + 100 - 169}{140} = -\frac{1}{7}.$$

$$\therefore \cos(180^\circ - C) = -\cos C = \frac{1}{7} = \cdot 1429,$$

correct to the fourth place.

But, from Table II,

$$\sin 8^\circ 10' = \cdot 1421$$

$$\text{diff. for } 2\tfrac{2}{3}' = \frac{8}{9} \times \text{diff. for } 3' \qquad = \qquad 8$$

$$\therefore \sin 8^\circ 12' 40'' = \cdot 1429.$$

$$\therefore \cos(180^\circ - C) = \sin 8^\circ 12' 40'' = \cos(90^\circ - 8^\circ 12' 40'') = \cos(81^\circ 47' 20'').$$

$$\therefore C = 180^\circ - 81^\circ 47' 20'' = 98^\circ 12' 40''.$$

This is the nearest result that can be obtained with four-figure tables. If seven-figure tables be used, the answer would be found to be $98^\circ 12' 48''$.

Ex. 2. *The sides of a triangle are* 32, 40, *and* 66 *feet; find the angle opposite the greatest side, having given that*

$$\log 207 = 2 \cdot 31597, \quad \log 1073 = 3 \cdot 03060,$$

$$L \cot 66^\circ 18' = 9 \cdot 64243, \textit{ tabulated difference for } 1' = 34.$$

Here $$a=32,\ b=40,\ \text{and}\ c=66,$$

so that $$s=\frac{32+40+66}{2}=69,\ s-a=37,\ s-b=29,\ \text{and}\ s-c=3.$$

Hence $$\cot\frac{C}{2}=\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=\sqrt{\frac{69\times 3}{37\times 29}}=\sqrt{\frac{207}{1073}}.$$

$$L\cot\frac{C}{2}=10+\frac{1}{2}[\log 207-\log 1073]$$

$$=10+1\cdot157985-1\cdot51530$$

$$=9\cdot642685.$$

$L\cot\frac{C}{2}$ is therefore *greater* than $L\cot 66°\,18'$,

so that $$\frac{C}{2}\ \text{is } \textit{less} \text{ than } 66°\,18'.$$

Let then $\frac{C}{2}=66°\,18'-x''$.

The difference in the logarithm corresponding to difference of x' in the angle therefore

$$\begin{aligned}=&\ \ 9\cdot642685\\ &-9\cdot64243\\ \hline =&\ \ \cdot000255.\end{aligned}$$

Also the difference for $60''=\cdot00034$.

Hence $$\frac{x}{60}=\frac{\cdot000255}{\cdot000340},$$

so that $$x=\frac{255}{340}\times 60=45.$$

$$\therefore\ \frac{C}{2}=66°\,18'-45''=66°\,17'\,15'',\ \text{and hence}\ C=132°\,34'\,30''.$$

EXAMPLES. XXIII.

[*The student should verify the results of some of the following examples* (*e.g.* *Nos.* 1, 7, 8, 12, 13, 14) *by an accurate graph.*]

1. If the sides of a triangle be 56, 65, and 33 feet, find the greatest angle.

2. The sides of a triangle are 7, $4\sqrt{3}$, and $\sqrt{13}$ yards respectively. Find the number of degrees in its smallest angle.

3. The sides of a triangle are x^2+x+1, $2x+1$, and x^2-1; prove that the greatest angle is 120°.

4. The sides of a triangle are a, b, and $\sqrt{a^2+ab+b^2}$ feet; find the greatest angle.

5. If $a=2$, $b=\sqrt{6}$, and $c=\sqrt{3}-1$, solve the triangle.

6. If $a=2$, $b=\sqrt{6}$, and $c=\sqrt{3}+1$, solve the triangle.

7. If $a=9$, $b=10$, and $c=11$, find B, given

$$\cos 58° 57' = \cdot 51579$$

and

$$\cos 59° = \cdot 51504.$$

8. The sides of a triangle are 2, 3, and 4; find the greatest angle, having given

$$\cos 75° 30' = \cdot 25038$$

and

$$\cos 75° 33' = \cdot 24954.$$

9. The sides of a triangle are 130, 123, and 77 feet. Find the greatest angle, having given

$$\log 2 = \cdot 30103, \quad L \tan 38° 39' = 9 \cdot 90294,$$

and

$$L \tan 38° 40' = 9 \cdot 90320.$$

10. Find the greatest angle of a triangle whose sides are 242, 188, and 270 feet, having given

$$\log 2 = \cdot 30103, \quad \log 3 = \cdot 4771213, \quad \log 7 = \cdot 8450980,$$

$$L \tan 38° 20' = 9 \cdot 8980104, \text{ and } L \tan 38° 19' = 9 \cdot 8977507.$$

Making use of the tables, find, to the nearest minute, the angles of the triangles in the following cases:

11. $a=5$, $b=4$ and $c=7$.

12. $a=9$ cms., $b=10$ cms. and $c=17$ cms.; check the result by an accurate drawing and measurement.

13. $a=25$, $b=26$, and $c=27$.

14. $a=17$, $b=20$, and $c=27$.

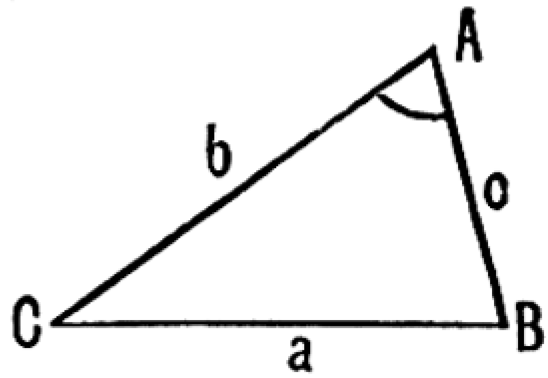
15. $a=2000$, $b=1050$, and $c=1150$.

16. $a=534$, $b=856$, and $c=610$.

17. If $a=638 \cdot 4$, $b=532 \cdot 7$, and $c=935 \cdot 1$ feet, find the angle A to the nearest minute.

118. *Case II. Given two sides b and c and the included angle A.*

Taking b to be the greater of the two given sides, we have



$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}\ \text{(Art. 107)}\ldots(1),$$

and
$$\frac{B+C}{2}=90^\circ-\frac{A}{2}\ \ldots\ldots(2).$$

These two relations give us

$$\frac{B-C}{2}\ \text{and}\ \frac{B+C}{2},$$

and therefore, by addition and subtraction, B and C.

The third side a is then known from the relation

$$\frac{a}{\sin A}=\frac{b}{\sin B},$$

which gives
$$a=b\,\frac{\sin A}{\sin B},$$

and thus determines a.

The side a may also be found from the formula

$$a^2=b^2+c^2-2bc\cos A.$$

This is not adapted to logarithmic calculation but is sometimes useful, especially when the sides a and b are small numbers.

119. Ex. 1. *If $b=\sqrt{3}$, $c=1$, and $A=30^\circ$, solve the triangle.*

We have
$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{\sqrt{3}-1}{\sqrt{3}+1}\cot 15^\circ.$$

Now
$$\tan 15^\circ=\frac{\sqrt{3}-1}{\sqrt{3}+1}\ \text{(Art. 55)},$$

so that
$$\cot 15^\circ=\frac{\sqrt{3}+1}{\sqrt{3}-1}.$$

Hence
$$\tan\frac{B-C}{2}=1.$$

$$\therefore\ \frac{B-C}{2}=45^\circ\ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also $$\frac{B+C}{2}=90^\circ-\frac{A}{2}=90^\circ-15^\circ=75^\circ\ldots\ldots\ldots\ldots(2).$$

By addition, $B=120^\circ$, and, by subtraction, $C=30^\circ$.

Since $A=C$, we have $a=c=1$.

Otherwise. We have

$$a^2=b^2+c^2-2bc\cos A=3+1-2\sqrt{3}\,.\,\frac{\sqrt{3}}{2}=1,$$

so that $$a=1=c.$$

$$\therefore\ C=A=30^\circ,$$

and $$B=180^\circ-A-C=120^\circ.$$

Ex. 2. *Using four figure tables, find the angles of the triangle in which $a=43$, $b=29$, and $C=43^\circ 14'$.*

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{14}{72}\cot 21^\circ 37'=\frac{7}{36}\tan 68^\circ 23'.$$

From Tables I and V,

$$\begin{array}{lr} & \log 7=\quad \cdot 8451 \\ & L\tan 68^\circ 20'=10\cdot 4009 \\ \text{diff. for} & 3'=\qquad 11 \\ \hline & 11\cdot 2471 \\ & \log 36=\ 1\cdot 5563 \\ \hline & \therefore\ L\tan\frac{A-B}{2}=\ 9\cdot 6908 \\ \text{but} & L\tan 26^\circ 8'=\ 9\cdot 6907 \\ \hline & 1. \end{array}\qquad \text{diff. for } 1'=3$$

$$\therefore\ \text{diff. for }\angle=\frac{1}{3}\times 1'=20''.$$

$$\left.\begin{array}{l}\therefore\ \dfrac{A-B}{2}=26^\circ 8' 20'' \\ \dfrac{A+B}{2}=90^\circ-\dfrac{C}{2}=68^\circ 23'\end{array}\right\}.$$

Hence by addition $$A=94^\circ 31' 20'',$$
and by subtraction $$B=42^\circ 14' 40''.$$

Ex. 3. *If $b=215$, $c=105$, and $A=74^\circ 27'$, find the remaining angles and also a third side a, having given*

$$\log 2=\cdot 30103,\qquad \log 11=1\cdot 04139,$$
$$\log 105=2\cdot 02119,\quad \log 212\cdot 48=2\cdot 32731,$$
$$L\cot 37^\circ 13'=10\cdot 11947,\ \textit{diff. for } 1'=26,$$
$$L\tan 24^\circ 20'=9\cdot 65535,\ \textit{diff. for } 1'=34,$$
$$L\sin 74^\circ 27'=9\cdot 98381,$$
and $$L\operatorname{cosec} 28^\circ 25'=10\cdot 32250,\ \textit{diff. for } 1'=23.$$

Here $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{11}{32}\cot 37^\circ 13' 30''.$

Now

$L\cot 37^\circ 13' =$	10·11947	
diff. for $30'' = -$	· 13	
$\therefore L\cot 37^\circ 13' 30'' =$	10·11934	
$\log 11 =$	1·04139	
	11·16073	·30103
$\log 32 =$	1·50515	5
$\therefore L\tan\frac{1}{2}(B-C) =$	9·65558.	1·50515.
But $L\tan 24^\circ 20' =$	9·65535	
diff. $=$	23	

$=$ diff. for $\frac{23}{34}$ of $60''$

$=$ diff. for $41''$.

$$\therefore \frac{B-C}{2}=24^\circ\, 20'\, 41''.$$

But $\frac{B+C}{2}=90^\circ-\frac{A}{2}=52^\circ\, 46'\, 30''.$

$\therefore$ by addition, $B=77^\circ\ \ 7'\, 11''$,

and, by subtraction, $C=28^\circ\, 25'\, 49''$.

Again $\frac{a}{\sin A}=\frac{c}{\sin C}=c\operatorname{cosec} C,$

$\therefore a=105\sin 74^\circ 27' \operatorname{cosec} 28^\circ 25' 49''.$

But $L\operatorname{cosec} 28^\circ 25' =$	10·32250	
diff. for $49'' = -$	· 19	$\frac{49}{60}\times 23$
$L\operatorname{cosec} 28^\circ 25' 49'' =$	10·32231	$=\frac{1127}{60}$
$L\sin 74^\circ 27' =$	9·98381	
$\log 105 =$	2·02119	$=19.$
	22·32731	
	20	
$\therefore \log a =$	2·32731	
$\therefore a =$	212·48.	

EXAMPLES. XXIV.

[*The student should verify the results of some of the following examples (e.g. Nos.* 6, 10, 11, 12, 13) *by an accurate graph.*]

1. If $a=2$, $b=1+\sqrt{3}$, and $C=60°$, solve the triangle.

2. One angle of a triangle is 30° and the lengths of the sides adjacent to it are 40 and $40\sqrt{3}$ yards. Find the length of the third side and the number of degrees in the other angles.

3. Two sides of a triangle are $\sqrt{3}+1$ and $\sqrt{3}-1$, and the included angle is 60°; find the other side and angles.

4. If $b=1$, $c=\sqrt{3}-1$, and $A=60°$, find the length of the side a.

5. If $b=91$, $c=125$ and $\tan\frac{A}{2}=\frac{17}{6}$, prove that $a=204$.

6. If $a=13$, $b=7$, and $C=60°$, find A and B, given that

$$\log 3=\cdot 47712,$$

and $$L\tan 27°\,27'=9\cdot 71555, \text{ diff. for } 1'=31.$$

7. If $a=21$, $b=11$, and $C=34°\,42'\,30''$, find A and B, given

$$\log 2=\cdot 30103,$$

and $$L\tan 72°\,38'\,45''=10\cdot 50515.$$

8. If $b=90$, $c=70$, and $A=72°48'30''$, find B and C, given

$$\cot 36°\,24'\,15''=1\cdot 35617,$$

$$\tan 9°\,37'=\cdot 16944,$$

and $$\tan 9°\,38'=\cdot 16974.$$

9. If the angles of a triangle be in A. P. and the lengths of the greatest and least sides be 24 and 16 feet respectively, find the length of the third side and the angles, given

$$\log 2=\cdot 30103,\ \log 3=\cdot 47712,$$

and $$L\tan 19°\,6'=9\cdot 53943, \text{ diff. for } 1'=41.$$

10. If $a=2b$, and $C=120°$, find the values of A, B, and the ratio of c to a, given that

$$\log 3=\cdot 4771213,$$

and $$L\tan 10°\,53'=9\cdot 2839070, \text{ diff. for } 1'=6808.$$

11. If $b=14$, $c=11$, and $A=60°$, find B and C, given that

$$\log 2 = \cdot 30103,\ \log 3 = \cdot 4771213,$$
$$L \tan 11° 44' = 9\cdot 3174299,$$
and
$$L \tan 11° 45' = 9\cdot 3180640.$$

In the following examples, the required logarithms must be taken from the tables.

12. If $a=30$, $b=20$ and $C=22°$, find the other angles.

13. If $b=6$, $c=4$ and $A=37°$, find the other angles.

14. If $b=27$, $c=43$ and $A=44°$, solve the triangle, finding the angles to the nearest minute.

15. If $b=29\cdot 6$ ft., $c=13\cdot 4$ ft. and $A=33° 36'$, find the other angles.

16. If $a=242\cdot 5$, $b=164\cdot 3$, and $C=54° 36'$, solve the triangle, finding the angles to the nearest minute.

17. If $b=130$, $c=63$, and $A=42° 16'$, solve the triangle, finding the angles to the nearest minute.

18. Two sides of a triangle being 2265 and 1779 feet, and the included angle $58° 18'$, find the remaining angles.

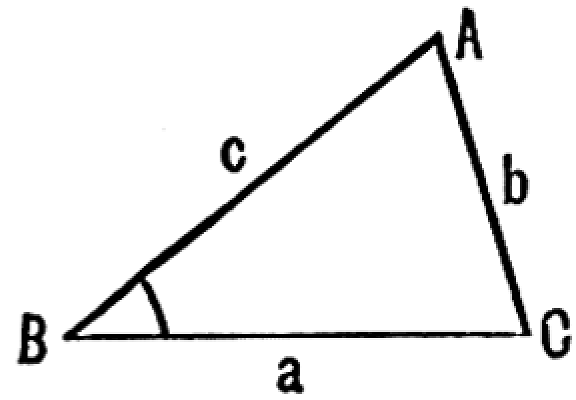
19. Two sides of a triangle being $237\cdot 1$ and 131 feet, and the included angle $57° 58'$, find the remaining angles.

120. *Case III. Given two sides b and c and the angle B opposite to one of them.*

The angle C is given by the relation

$$\frac{\sin C}{c} = \frac{\sin B}{b},$$

i.e.
$$\sin C = \frac{c}{b} \sin B \quad \text{......(1).}$$



Taking logarithms, we determine C, and then $A\ (=180° - B - C)$ is found.

The remaining side a is then found from the relation

$$\frac{a}{\sin A} = \frac{b}{\sin B},$$

i.e.
$$a = b\frac{\sin A}{\sin B} \quad \text{.....................(2).}$$

121. The equation (1) of the previous article gives in some cases no value, in some cases one, and sometimes two values, for C.

Hence, for some values of b, c and B, there is a doubt or ambiguity in the determination of the triangle; this case is therefore called the **Ambiguous Case** of the solution of triangles.

Suppose that, given the elements b, c, and B, we construct, or attempt to construct, the triangle.

We first measure an angle ABD equal to the given angle B.

We then measure along BA a distance BA equal to the given distance c, and thus determine the angular point A.

We have now to find a third point C, which must lie on BD and must also be such that its distance from A shall be equal to b.

To obtain it, we describe with centre A a circle whose radius is b.

The point or points, if any, in which this circle meets BD will determine the position of C.

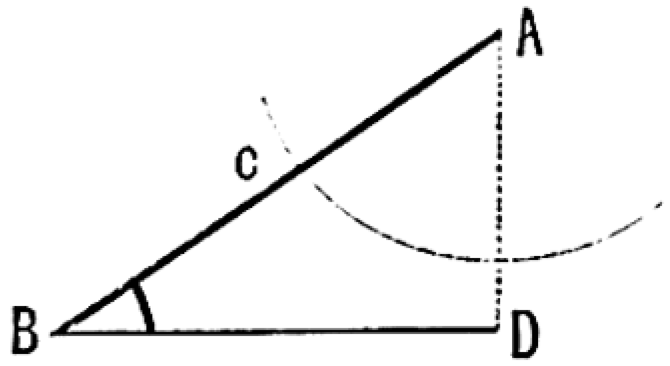


Fig. 1

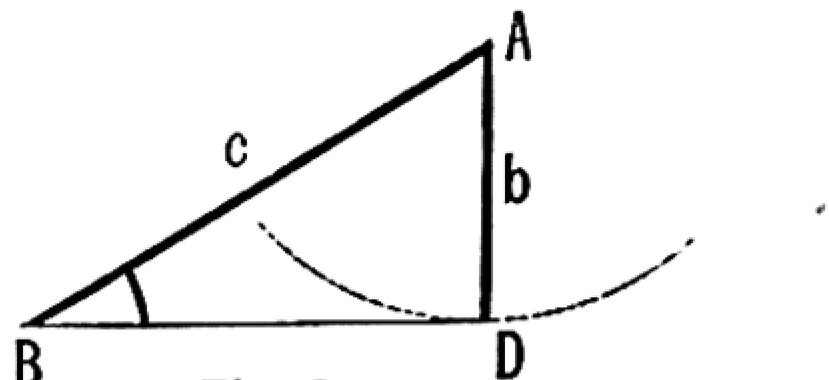


Fig. 2

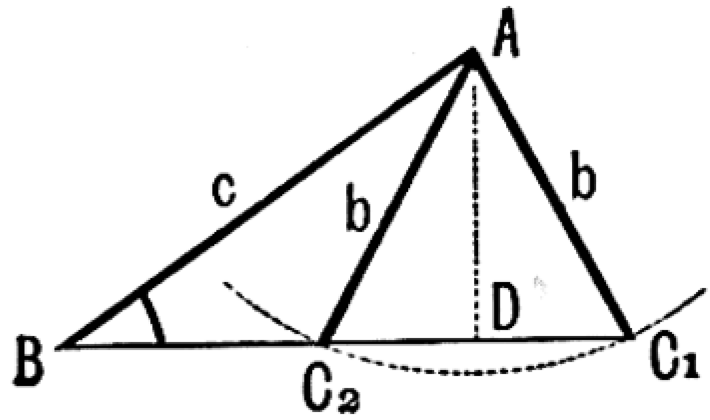


Fig. 3

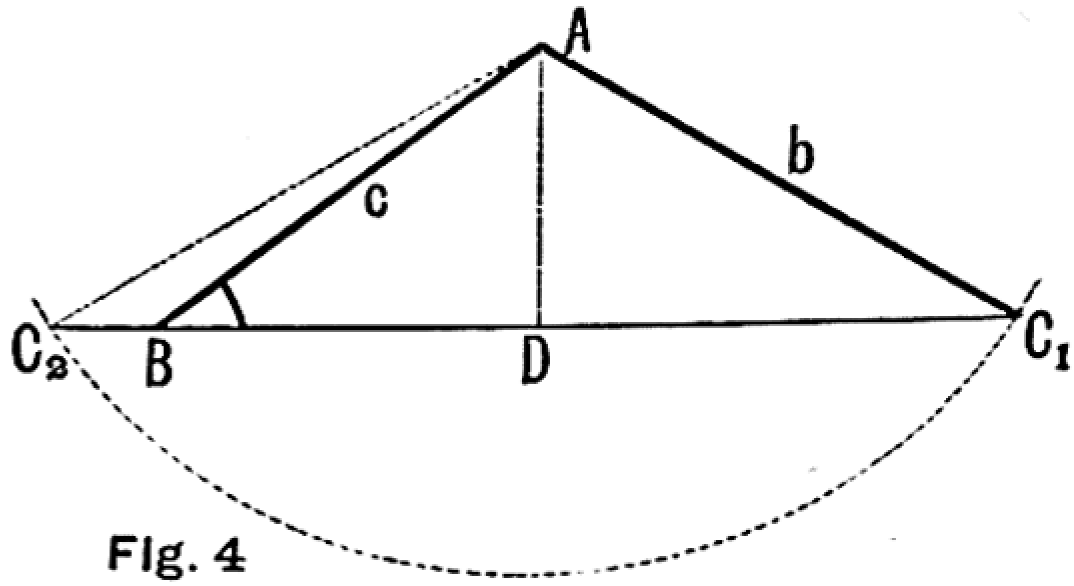


Fig. 4

Draw AD perpendicular to BD, so that

$$AD = AB \sin B = c \sin B.$$

One of the following events will happen.

The circle may not reach BD (Fig. 1) or it may touch BD (Fig. 2), or it may meet BD in two points C_1 and C_2 (Figs. 3 and 4).

In the case of Fig. 1, it is clear that there is no triangle satisfying the given condition.

Here $\qquad b < AD$, *i.e.* $< c \sin B$.

In the case of Fig. 2, there is one triangle ABD which is right-angled at D. Here

$$b = AD = c \sin B.$$

In the case of Fig. 3, there are two triangles ABC_1 and ABC_2. Here b lies in magnitude beween AD and c, *i.e.* b is $> c \sin B$ and $< c$.

In the case of Fig. 4, there is only one triangle ABC_1 satisfying the given conditions [the triangle ABC_2 is inadmissible; for its angle at B is not equal to B but is equal to $180° - B$]. Here b is greater than both $c \sin B$ and c.

In the case when B is obtuse, the proper figures should be drawn. It will then be seen that when $b < c$ there is no triangle (for in the corresponding triangles ABC_1 and ABC_2 the angle at B will be $180° - B$ and not B). If $b > c$, it will be seen that there is one triangle, and only one, satisfying the given conditions.

To sum up:

Given the elements b, c, and B of a triangle,

(α) If b be $< c \sin B$, there is no triangle.

(β) If $b = c \sin B$, there is one triangle right-angled.

(γ) If b be $> c \sin B$ and $< c$ and B be acute, there are two triangles satisfying the given conditions.

(δ) If b be $> c$, there is only one triangle.

Clearly if $b = c$, the points B and C_2 in Fig. 3 coincide and there is only one triangle.

(ϵ) If B be obtuse, there is no triangle except when $b > c$.

122. Without determining the other angles the third side may be found thus.

From the figure of Art. 120, we have

$$b^2 = c^2 + a^2 - 2ca \cos B.$$

Therefore

$$a^2 - 2ac \cos B + c^2 \cos^2 B = b^2 - c^2 + c^2 \cos^2 B = b^2 - c^2 \sin^2 B.$$

$$\therefore \quad a - c \cos B = \pm \sqrt{b^2 - c^2 \sin^2 B},$$

i.e.

$$a = c \cos B \pm \sqrt{b^2 - c^2 \sin^2 B} \quad \ldots\ldots\ldots\ldots(1).$$

Now (1) is an equation giving two values for a when b, c, and B are given.

It would be found that as in the previous article these two values of a would be real and positive only when $b > c \sin B$ and $< c$.

123. Ex. *Given $b = 16$, $c = 25$, and $B = 33° 15'$, prove that the triangle is ambiguous and find the other angles, having given*

$$\log 2 = \cdot 30103, \quad L \sin 33° 15' = 9 \cdot 73901,$$

$$L \sin 58° 56' = 9 \cdot 93276,$$

and

$$L \sin 58° 57' = 9 \cdot 93284.$$

We have

$$\sin C = \frac{c}{b} \sin B = \frac{25}{16} \sin B = \frac{100}{64} \sin B = \frac{10^2}{2^6} \sin 33° 15'.$$

Hence

$$L \sin C = 2 + L \sin 33° 15' - 6 \log 2$$
$$= 9 \cdot 93283.$$

Hence

$L \sin C = 9 \cdot 93283$	$L \sin 58° 57' = 9 \cdot 93284$
$L \sin 58° 56' = 9 \cdot 93276$	$L \sin 58° 56' = 9 \cdot 93276$
Diff. $=$ 7	Diff. for $1' =$ 8

$$\therefore \text{ angular diff.} = \frac{7}{8} \times 60''$$
$$= 53'' \text{ nearly.}$$

$$\therefore \quad C = 58° 56' 53'' \text{ or } 180° - 58° 56' 53''.$$

Hence (Fig. 3, Art. 121) we have

$$C_1 = 58° 56' 53'', \text{ and } C_2 = 121° 3' 7''.$$

$$\therefore \angle BAC_1 = 180° - 33° 15' - 58° 56' 53'' = 87° 48' 7'',$$

and

$$\angle BAC_2 = 180° - 33° 15' - 121° 3' 7'' = 25° 41' 53''.$$

EXAMPLES. XXV.

[*The student should verify the results of some of the following examples (e.g. Nos.* 3, 5, 7, 8, 9, 11, 12, 13) *by an accurate graph.*]

1. If $a=5$, $b=7$, and $\sin A=\frac{3}{4}$, is there any ambiguity?

2. If $a=2$, $c=\sqrt{3}+1$, and $A=45°$, solve the triangle.

3. If $a=100$, $c=100\sqrt{3}$, and $A=30°$, solve the triangle.

4. If $2b=3a$, and $\tan^2 A=\frac{3}{5}$, prove that there are two values to the third side, one of which is double the other.

5. If $A=30°$, $b=8$, and $a=6$, find c.

6. In the ambiguous case given a, b, and A, prove that the difference between the two values of c is $2\sqrt{a^2-b^2\sin^2 A}$.

7. If $a=5$, $b=4$, and $A=45°$, find the other angles, having given

$$\log 2=\cdot 30103,\quad L\sin 33°\,29'=9\cdot 75205,$$

and

$$L\sin 33°\,30'=9\cdot 75310.$$

8. Point out whether or no the solutions of the following triangles are ambiguous.

Find the smaller value of the third side in the ambiguous case and the other angles in both cases.

(1) $A=30°$, $c=250$ feet, and $a=125$ feet;

(2) $A=30°$, $c=250$ feet, and $a=200$ feet.

Given

$$\sin 38°\,41'=\cdot 625,$$

and

$$\sin 8°\,41'=\cdot 15097.$$

9. If $a=9$, $b=12$, and $A=30°$, solve the resulting triangles, having given

$$\log 2=\cdot 30103,\qquad \log 3=\cdot 47712,$$
$$\log 171=2\cdot 23301,\qquad \log 368=2\cdot 56635,$$
$$L\sin 11°\,48'\,39''=9\cdot 31108,\quad L\sin 41°\,48'\,39''=9\cdot 82391,$$

and

$$L\sin 108°\,11'\,21''=9\cdot 97774.$$

10. Two straight roads intersect at an angle of 30°; from the point of junction two pedestrians A and B start at the same time, A walking along one road at the rate of 5 miles per hour and B walking uniformly along the other road. At the end of 3 hours they are 9 miles apart. Shew that there are two rates at which B may walk to fulfil this condition and find them.

For the following examples, a book of tables will be required; for the first three the angles are to be found to the nearest minute.

11. Given $a=7$, $b=4$, $B=30°$, solve the triangles.

12. Given $b=7$, $c=3$, $B=30°$, solve the triangle.

13. Given $a=5$, $c=10$, $A=20°$, solve the triangles.

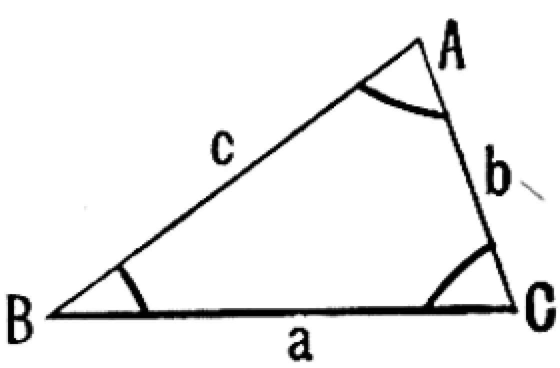
14. Two sides of a triangle are 1015 feet and 732 feet, and the angle opposite the latter side is 40°; find the angle opposite the former and prove that more than one value is admissible.

15. Two sides of a triangle being 5374 and 1586 feet, and the angle opposite the latter being 15° 11′, calculate the other angles of the triangle or triangles.

16. Given $A=10°$, $a=2308$, and $b=7903$, find the smaller value of c.

124. *Case IV. Given one side and two angles, viz. a, B, and C.*

Since the three angles of a triangle are together equal to two right angles, the third angle is given also.



The sides b and c are now obtained from the relations

$$\frac{b}{\sin B} = \frac{c}{\sin C} = \frac{a}{\sin A},$$

giving $$b = a\frac{\sin B}{\sin A}, \text{ and } c = a\frac{\sin C}{\sin A}.$$

125. *Case V. The three angles A, B, and C given.*

Here the ratios only of the sides can be determined by the formulae

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Their absolute magnitudes cannot be found.

EXAMPLES. XXVI.

1. If $\cos A = \frac{17}{22}$ and $\cos C = \frac{1}{14}$, find the ratio of $a : b : c$.

2. If $\cos A = \frac{3}{5}$ and $\cos B = \frac{33}{65}$, prove that the sides of the triangle are proportional to 13, 14 and 15.

3. If $A = 45°$, $B = 75°$, and $C = 60°$, prove that $a + c\sqrt{2} = 2b$.

4. Two angles of a triangle are $41° 13'$ and $71° 19'$ and the side opposite the first angle is 55; find the side opposite the latter angle, given

$$\log 55 = 1{\cdot}74036, \ \log 7907 = 3{\cdot}89803,$$
$$L \sin 41° 13' = 9{\cdot}81882,$$
and
$$L \sin 71° 19' = 9{\cdot}97649.$$

5. From each of two ships, one mile apart, the angle is observed which is subtended by the other ship and a beacon on shore; these angles are found to be $52° 25' 15''$ and $75° 9' 30''$ respectively. Given

$$L \sin 75° 9' 30'' = 9{\cdot}9852635,$$
$$L \sin 52° 25' 15'' = 9{\cdot}8990055, \ \log 1{\cdot}2197 = {\cdot}0862530,$$
and
$$\log 1{\cdot}2198 = {\cdot}0862886,$$
find the distance of the beacon from each of the ships.

6. The base angles of a triangle are $22\frac{1}{2}°$ and $112\frac{1}{2}°$; prove that the base is equal to twice the height.

For the following questions a book of tables is required.

7. The base of a triangle being seven feet and the base angles $129° 23'$ and $38° 36'$, find the length of its shorter side.

8. If the angles of a triangle be as 5 : 10 : 21, and the side opposite the smaller angle be 3 feet, find the other sides.

9. The angles of a triangle being 150°, $18° 20'$, and $11° 40'$, and the longest side being 1000 feet, find the length of the shortest side.

10. Two angles of a triangle are $61° 25'$ and $43° 37'$ and the side opposite the former is 637 feet; find the side opposite the latter.

11. The base of a triangle is 2 feet long and the angles adjacent to the base are $30°$ and $45°$; find the lengths of the three perpendiculars of the triangle.

12. To get the distance of a point A from a point B, a line BC and the angles ABC and BCA are measured, and are found to be 287 yards and $55° 32'$ and $51° 8'$ respectively. Find the distance AB.

13. To find the distance from A to P a distance, AB, of 1000 yards is measured in a convenient direction. At A the angle PAB is found to be $41° 18'$ and at B the angle PBA is found to be $114° 38'$. What is the required distance to the nearest yard?

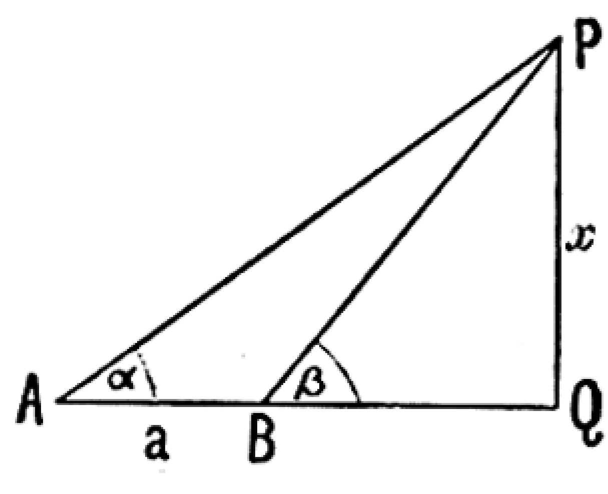
CHAPTER XII.

HEIGHTS AND DISTANCES.

126. IN the present chapter we shall consider some questions of the kind which occur in land-surveying. Simple questions of this kind have already been considered in Chapter III.

127. *To find the height of an inaccessible tower by means of observations made at distant points.*

Suppose PQ to be the tower and that the ground passing through the foot Q of the tower is horizontal. At a point A on this ground measure the angle of elevation α of the top of the tower.



Measure off a distance $AB\ (=a)$ from A directly toward the foot of the tower, and at B measure the angle of elevation β.

To find the unknown height x of the tower, we have to connect it with the measured length a. This is best done as follows:

From the triangle PBQ, we have

$$\frac{x}{BP} = \sin\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and, from the triangle PAB, we have

$$\frac{PB}{a} = \frac{\sin PAB}{\sin BPA} = \frac{\sin \alpha}{\sin(\beta - \alpha)} \qquad \text{.............(2)},$$

since $\angle BPA = \angle QBP - \angle QAP = \beta - \alpha$.

From (1) and (2), by multiplication, we have

$$\frac{x}{a} = \frac{\sin \alpha \sin \beta}{\sin(\beta - \alpha)},$$

i.e.

$$x = a \frac{\sin \alpha \sin \beta}{\sin(\beta - \alpha)}.$$

The height x is therefore given in a form suitable for logarithmic calculation.

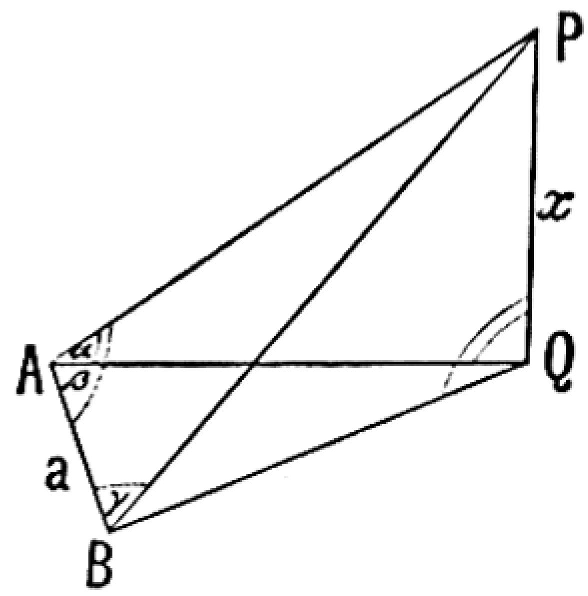
Numerical Example. If $a = 100$ feet, $\alpha = 30°$, and $\beta = 60°$, then

$$x = 100 \frac{\sin 30° \sin 60°}{\sin 30°} = 100 \times \frac{\sqrt{3}}{2} = 86{\cdot}6 \text{ feet.}$$

128. It is often not convenient to measure AB directly towards Q.

Measure therefore AB in any other suitable direction on the horizontal ground, and at A measure the angle of elevation α of P, and also the angle PAB $(= \beta)$.

At B measure the angle PBA $(= \gamma)$.



In the triangle PAB, we have then

$$\angle APB = 180° - \angle PAB - \angle PBA = 180° - (\beta + \gamma).$$

Hence

$$\frac{AP}{a} = \frac{\sin PBA}{\sin BPA} = \frac{\sin \gamma}{\sin(\beta + \gamma)}.$$

From the triangle PAQ, we have

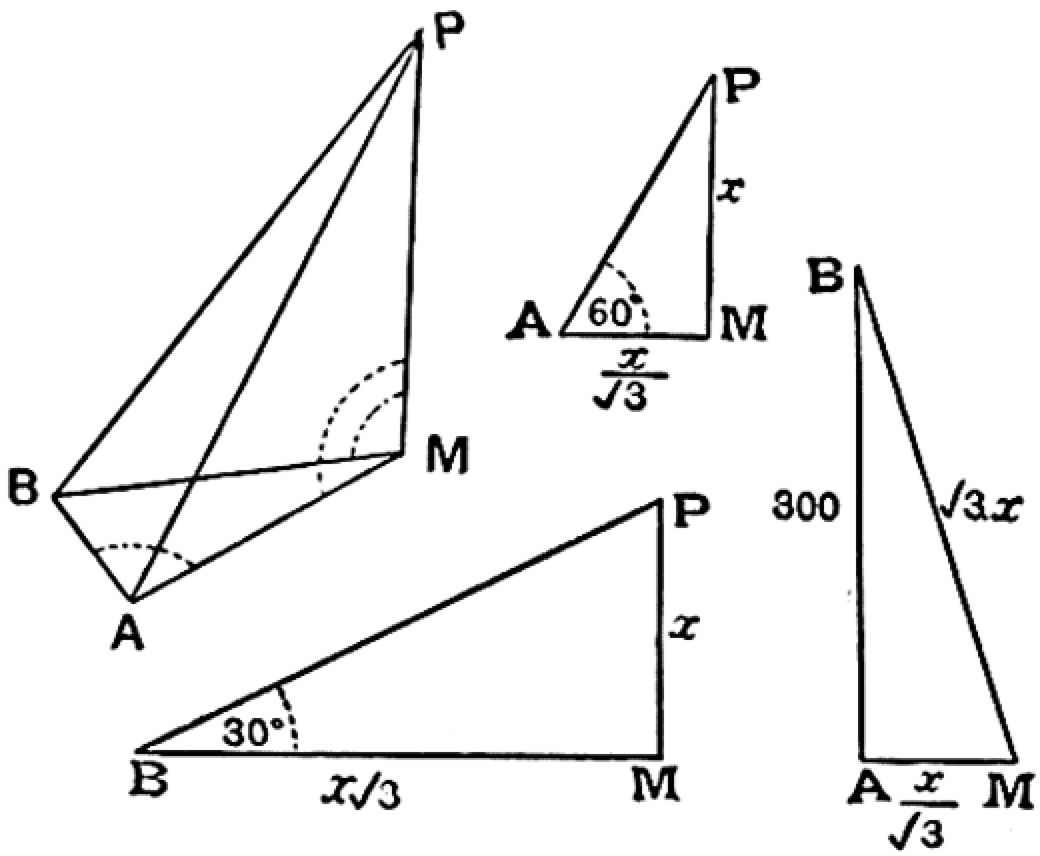
$$x = AP \sin \alpha = a \frac{\sin \alpha \sin \gamma}{\sin(\beta + \gamma)}.$$

Hence x is found by an expression suitable for logarithmic calculation.

129. Instead of the measurements in the foregoing article the elevations, PAQ and PBQ, at the points A and B might be taken and the distance AB and the angle QAB measured. Unless, however, the angle QAB is a right angle the solution is not in general easy or suitable for logarithmic calculation.

When QAB is a right angle, the method of procedure will be easily seen from the following example.

A man observes that at a point due south of a certain tower its angle of elevation is 60°; he then walks 300 feet due west on a horizontal plane and finds that the angle of elevation is 30°; find the height of the tower and his original distance from it.



Let P be the top, and PM the height, of the tower, A the point due south of the tower and B the point due west of A.

The angles PMA, PMB, and MAB are therefore all right angles.

For simplicity, since the triangles PAM, PBM, and ABM are in different planes, they are reproduced in the second, third, and fourth figures and drawn to scale.

We are given $AB = 300$ feet, $\angle PAM = 60°$, and $\angle PBM = 30°$.

Let the height of the tower be x feet.

From the second figure,

$$\frac{AM}{x} = \cot 60° = \frac{1}{\sqrt{3}},$$

so that

$$AM = \frac{x}{\sqrt{3}}.$$

From the third figure,

$$\frac{BM}{x} = \cot 30° = \sqrt{3},$$

so that $$BM = \sqrt{3} \,.\, x.$$

From the last figure, we have

$$BM^2 = AM^2 + AB^2,$$

i.e. $$3x^2 = \frac{1}{3}x^2 + 300^2.$$

$$\therefore\ 8x^2 = 3 \times 300^2.$$

$$\therefore\ x = \frac{300\sqrt{3}}{2\sqrt{2}} = 150 \,.\, \frac{\sqrt{6}}{2} = 75 \times \sqrt{6}$$

$$= 75 \times 2{\cdot}44949 \ldots = 183{\cdot}71 \ldots \text{ feet.}$$

Also his original distance from the tower

$$= x \cot 60° = \frac{x}{\sqrt{3}} = 75 \times \sqrt{2}$$

$$= 75 \times (1{\cdot}4142 \ldots) = 106{\cdot}065 \ldots \text{ feet.}$$

More generally. If the angles PAM, PBM be any angles α and β and the distance AB be a, then on drawing similar figures to the above we have from the triangles PAM, PBM

$$AM = x \cot PAM = x \cot \alpha, \text{ and } BM = x \cot PBM = x \cot \beta.$$

Hence the triangle BAM gives

$$x^2 \cot^2 \beta = x^2 \cot^2 \alpha + a^2.$$

$$\therefore\ x^2 = \frac{a^2}{\cot^2 \beta - \cot^2 \alpha} = \frac{a^2 \sin^2 \alpha \sin^2 \beta}{\sin^2 \alpha \cos^2 \beta - \cos^2 \alpha \sin^2 \beta}$$

$$= \frac{a^2 \sin^2 \alpha \sin^2 \beta}{[\sin \alpha \cos \beta + \cos \alpha \sin \beta][\sin \alpha \cos \beta - \cos \alpha \sin \beta]}$$

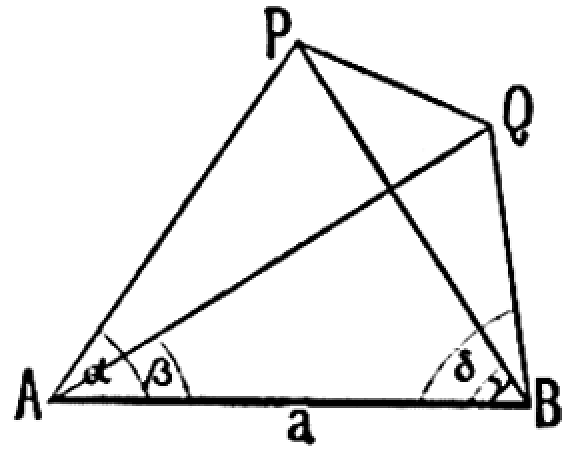
$$= \frac{a^2 \sin^2 \alpha \sin^2 \beta}{\sin (\alpha + \beta) \,.\, \sin (\alpha - \beta)}.$$

$$\therefore\ x = \frac{a \sin \alpha \sin \beta}{\sqrt{\sin (\alpha + \beta) \sin (\alpha - \beta)}};$$

this is a form suitable for logarithmic calculation.

130. *To find the distance between two inaccessible points by means of observations made at two points the distance between which is known, all four points being supposed to be in one plane.*

Let P and Q be two points whose distance apart, PQ is required.



Let A and B be the two known points whose distance apart, AB, is given to be equal to a.

At A measure the angles PAB and QAB, and let them be α and β respectively.

At B measure the angle PBA and QBA, and let them be γ and δ respectively.

Then in the triangle PAB we have one side a and the two adjacent angles α and γ given, so that, as in Art. 99, we have AP given by the relation

$$\frac{AP}{a}=\frac{\sin\gamma}{\sin APB}=\frac{\sin\gamma}{\sin(\alpha+\gamma)}\text{...........(1)},$$

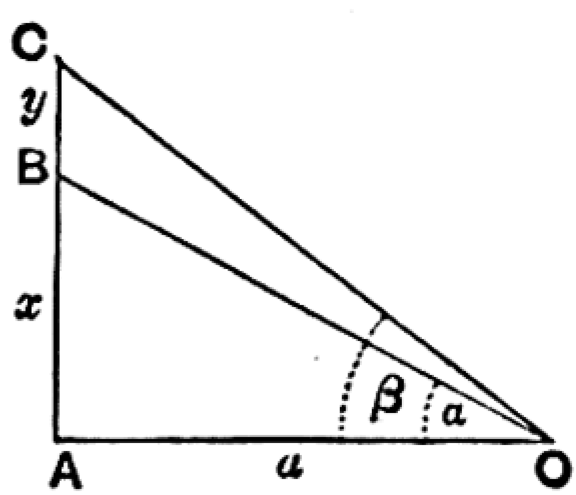
since $$\angle APB=180^\circ-\angle PAB-\angle PBA$$
$$=180^\circ-(\alpha+\gamma).$$

In the triangle QAB we have, similarly,

$$\frac{AQ}{a}=\frac{\sin\delta}{\sin AQB}=\frac{\sin\delta}{\sin(\beta+\delta)}\text{...........(2)}.$$

In the triangle APQ we have now determined the sides AP and AQ; also the included angle $PAQ\,(=\alpha-\beta)$ is known. We can therefore find the side PQ by the method of Art. 118.

131. Ex. *A flagstaff stands on the top of a tower; at a point distant a feet from the foot of the tower, the angular elevations of the top of the tower and flagstaff are α and β; find the height of the flagstaff.*



Let $AB\,(=x)$ be the height of the tower, and $BC\,(=y)$ the flagstaff; also let O be the observer.

Then $$x=a\tan\alpha\text{ ....(1)},$$

and $$x+y=a\tan\beta\text{ ....(2)}.$$

By subtraction,
$$y = a(\tan\beta - \tan\alpha)$$
$$= a\frac{\sin\beta\cos\alpha - \cos\beta\sin\alpha}{\cos\beta\cos\alpha},$$
$$\text{i.e. } y = a\frac{\sin(\beta-\alpha)}{\cos\beta\cos\alpha} \quad \text{......................(3).}$$

This latter form is suitable for logarithmic calculation.

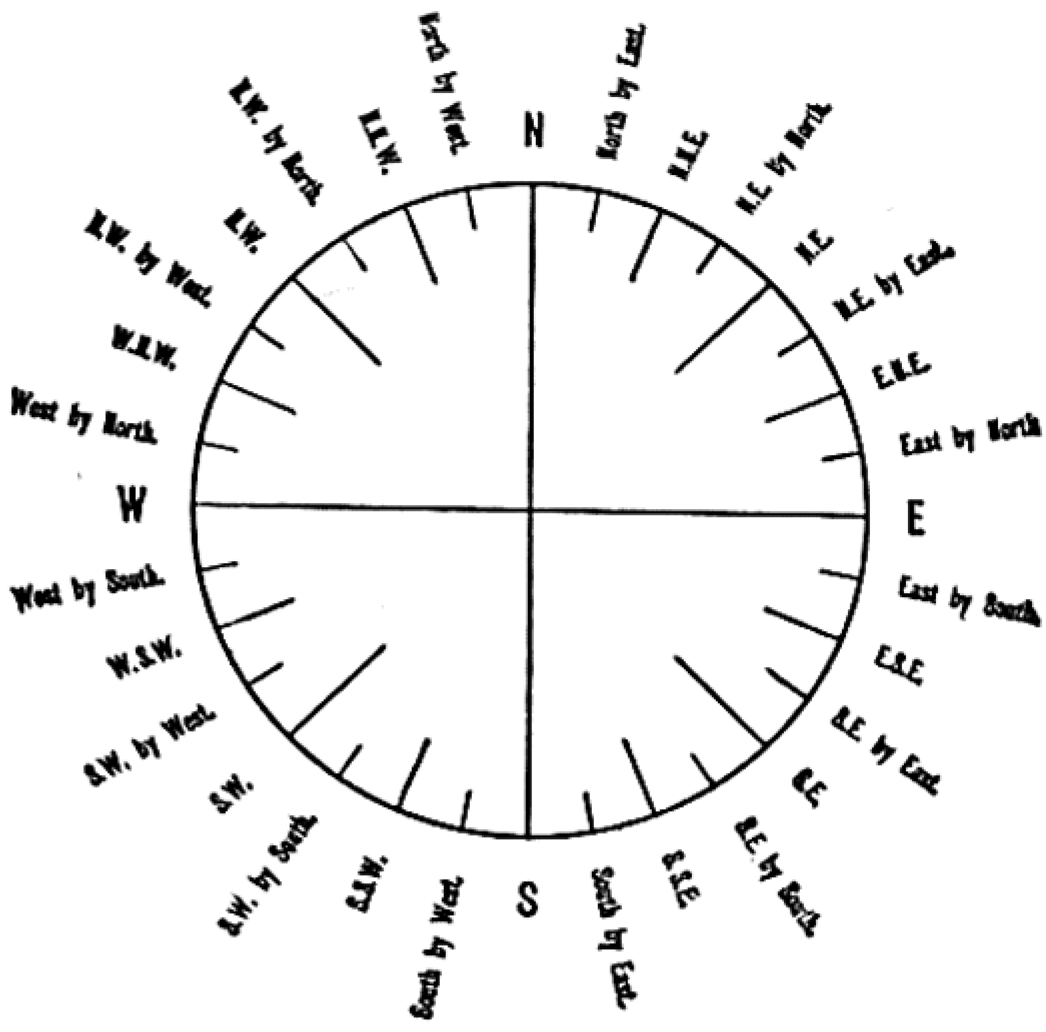
If again the height of the tower is given but not a, then (1) and (2) give, by division,
$$\frac{y+x}{x} = \frac{\tan\beta}{\tan\alpha} = \frac{\sin\beta\cos\alpha}{\cos\beta\sin\alpha},$$
$$\therefore\ y = x\left[\frac{\sin\beta\cos\alpha - \cos\beta\sin\alpha}{\cos\beta\sin\alpha}\right] = x\frac{\sin(\beta-\alpha)}{\cos\beta\sin\alpha}.$$

132. Bearings and Points of the Compass. The Bearing of a given point B as seen from a given point O is the direction in which B is seen from O. Thus if the direction of OB bisect the angle between East and North, the bearing of B is said to be North-East.

If a line is said to bear 20° West of North (for brevity written N. 20° W.), we mean that it is inclined to the North direction at an angle of 20°, this angle being measured from the North towards the West.

So W. 15° S. means a direction obtained as follows: first look West and then turn your eyes through an angle 15° towards the South and the direction you are now looking in is the required direction.

To facilitate the statement of the bearing of a point the circumference of the mariner's compass-card is divided into 32 equal portions, as in the above figure, and the subdivisions marked as indicated. Consider only the quadrant between East and North. The middle point of the arc between N. and E. is marked North-East (N.E.). The bisectors of the arcs between N.E. and N. and E. are respectively called North-North-East and East-North-East (N.N.E. and E.N.E.). The other four subdivisions, reckoning from N., are called North by East, N.E. by North, N.E. by East, and East by North. Similarly the other three quadrants are subdivided.



It is clear that the arc between two subdivisions of the card subtends an angle of $\frac{360°}{32}$, *i.e.* $11\frac{1}{4}°$, at the centre *O*.

EXAMPLES. XXVII.

1. A flagstaff stands on the middle of a square tower. A man on the ground, opposite the middle of one face and distant from it 100 feet, just sees the flag; on his receding another 100 feet, the tangents of elevation of the top of the tower and the top of the flagstaff are found to be $\frac{1}{2}$ and $\frac{5}{9}$. Find the dimensions of the tower and the height of the flagstaff, the ground being horizontal.

2. A man, walking on a level plane towards a tower, observes that at a certain point the angular height of the tower is 10°, and, after going 50 yards nearer the tower, the elevation is found to be 15°. Having given

$$L\sin 15° = 9{\cdot}4130, \quad L\cos 5° = 9{\cdot}9983,$$

and

$$\log 25{\cdot}78 = 1{\cdot}4113,$$

find the height of the tower in yards.

3. A person on a ship sailing north sees two lighthouses, which are 6 miles apart, in a line due west; after an hour's sailing one of them bears S.W. and the other S.S.W. Find the ship's rate.

4. A tower, 50 feet high, stands on the top of a mound; from a point on the ground the angles of elevation of the top and bottom of the tower are found to be 75° and 45° respectively; find the height of the mound.

5. A vertical pole (more than 100 feet high) consists of two parts, the lower being $\frac{1}{3}$rd of the whole. From a point in a horizontal plane through the foot of the pole and 40 feet from it, the upper part subtends an angle whose tangent is $\frac{1}{2}$. Find the height of the pole.

6. A person in a balloon, which has ascended vertically from flat land at the sea level, observes the angle of depression of a ship at anchor to be 30°; after descending vertically for 600 feet, he finds the angle of depression to be 15°; find the horizontal distance of the ship from the point of ascent.

7. A lighthouse, facing north, sends out a fan-shaped beam of light extending from north-east to north-west. An observer on a steamer, sailing due west, first sees the light when he is 5 miles away from the lighthouse and continues to see it for $30\sqrt{2}$ minutes. What is the speed of the steamer?

8. A person on a ship sees a lighthouse N.W. of himself. After sailing for 12 miles in a direction 15° south of W. the lighthouse is seen due N. Find the distance of the lighthouse from the ship in each position.

9. A tower stood at the foot of an inclined plane whose inclination to the horizon was 9°. A line 100 feet in length was measured straight up the incline from the foot of the tower, and at the end of this line the tower subtended an angle of 54°. Find the height of the tower, having given

$$\log 2 = \cdot 30103, \quad \log 114\cdot 4123 = 2\cdot 0584726,$$

and $$L \sin 54° = 9\cdot 9079576.$$

10. A man stands at a point X on the bank XY of a river with straight and parallel banks and observes that the line joining X to a point Z on the opposite bank makes an angle of 30° with XY. He then goes along the bank a distance of 200 yards to Y and finds that the angle ZYX is 60°. Find the breadth of the river.

11. From a point A on a level plane the angle of elevation of a balloon is a, the balloon being south of A; from a point B, which is at a distance c south of A, the balloon is seen northwards at an elevation of β; find the distance of the balloon from A and its height above the ground.

12. P is the top and Q the foot of a tower standing on a horizontal plane. A and B are two points on this plane such that AB is 32 feet and QAB is a right angle. It is found that $\cot PAQ = \frac{2}{5}$ and

$$\cot PBQ = \frac{3}{5};$$

find the height of the tower.

13. PQ is a tower standing on a horizontal plane, Q being its foot; A and B are two points on the plane such that the $\angle QAB$ is $90°$, and AB is 40 feet. It is found that

$$\tan PAQ = \frac{10}{3} \text{ and } \tan PBQ = 2.$$

Find the height of the tower.

14. The elevation of a steeple at a place due south of it is 45° and at another place due west of the former place the elevation is 15°. If the distance between the two places be a, prove that the height of the steeple is

$$\frac{a(\sqrt{3}-1)}{2\sqrt[4]{3}}.$$

15. A and B are two stations 1000 feet apart; P and Q are two stations in the same plane as AB and on the same side of it; the angles PAB, PBA, QAB, and QBA are respectively 75°, 30°, 45°, and 90°; find how far P is from Q and how far each is from A and B.

16. At the foot of a mountain the elevation of its summit is found to be 45°; after ascending one mile up a slope of 30° inclination the elevation is found to be 60°. Find the height of the mountain.

17. A vertical tower stands on a slope which is inclined at 15° to the horizon. From the foot of the tower a man ascends the slope for 80 feet, and then finds that the tower subtends an angle of 30°. Prove that the height of the tower is $40(\sqrt{6}-\sqrt{2})$ feet.

18. The altitude of a certain rock is 47°, and after walking towards it 1000 feet up a slope inclined at 30° to the horizon an observer finds its altitude to be 77°. Find the vertical height of the rock above the first point of observation, given that $\sin 4° = \cdot 0698$.

19. Two flagstaffs stand on a horizontal plane. A and B are two points on the line joining the bases of the flagstaffs and between them. The angles of elevation of the tops of the flagstaffs as seen from A are 30° and 60° and, as seen from B, they are 60° and 45°. If the length AB be 30 feet, find the heights of the flagstaffs and the distance between them.

20. A square tower stands upon a horizontal plane. From a point in this plane, from which three of its upper corners are visible, their angular elevations are respectively 45°, 60°, and 45°. Shew that the height of the tower is to the breadth of one of its sides as $\sqrt{6}(\sqrt{5}+1)$ to 4.

21. A man, walking due north, observes that the elevation of a balloon, which is due east of him and is sailing toward the north-west, is then 60°; after he has walked 400 yards the balloon is vertically over his head; find its height supposing it to have always remained the same.

22. A tower subtends an angle α at a point on the same level as the foot of the tower, and at a second point, h feet above the first, the depression of the foot of the tower is β. Find the height of the tower.

23. A column is E.S.E. of an observer, and at noon the end of the shadow is North-East of him. The shadow is 80 feet long and the elevation of the column at the observer's station is 45°. Find the height of the column.

24. A bridge has 5 equal spans, each of 100 feet measured from the centre of the piers, and a boat is moored in a line with one of the middle piers. The whole length of the bridge subtends a right angle as seen from the boat. Prove that the distance of the boat from the bridge is $100\sqrt{6}$ feet.

25. A ladder placed at an angle of 75° with the ground just reaches the sill of a window at a height of 27 feet above the ground on one side of a street. On turning the ladder over without moving its foot, it is found that when it rests against a wall on the other side of the street it is at an angle of 15° with the ground. Prove that the breadth of the street and the length of the ladder are respectively

$$27(3-\sqrt{3}) \text{ and } 27(\sqrt{6}-\sqrt{2}) \text{ feet.}$$

26. A rod of given length can turn in a vertical plane passing through the sun, one end being fixed on the ground; find the longest shadow it can cast on the ground.

Calculate the altitude of the sun when the longest shadow it can cast is $3\frac{1}{2}$ times the length of the rod.

27. Two stations due south of a tower, which leans towards the north, are at distances a and b from its foot; if α and β be the elevations of the top of the tower from these stations, prove that its inclination to the horizontal is the angle whose cotangent is

$$\frac{b \cot \alpha - a \cot \beta}{b - a}.$$

28. A tower 150 feet high stands on the top of a cliff 80 feet high. At what point on the plane passing through the foot of the cliff must an observer place himself so that the tower and the cliff may subtend equal angles, the height of his eye being 5 feet?

29. The angle of elevation of a cloud from a point h feet above a lake is α, and the angle of depression of its reflexion in the lake is β; prove that its height is $h\dfrac{\sin(\beta+\alpha)}{\sin(\beta-\alpha)}$.

EXAMPLES XXVIII.

[*For the following examples a book of tables will be wanted or the four-figure tables at the end of this book may be used as an approximation. Some of these examples should be verified by a graph.*]

1. To find the breadth AB of a river an observer measures in AB produced a length BC of 20 yards, and then walks a distance CP equal to 100 yards at right angles to AC. He then finds that AC subtends an angle 35° 40′ at P. Find the breadth of the river and the angle that BC subtends at P.

2. A person standing at the edge of a river observes the altitude of the top of a tower on the edge of the opposite side to be 55°; receding backwards 30 feet he finds it to be 48°. Find the breadth of the river.

3. At a point on a horizontal plane the elevation of the summit of a mountain is found to be 22° 15′, and at another point on the plane, a mile further away in a direct line, its elevation is 10° 12′; find the height of the mountain.

4. From the top of a hill the angles of depression of two successive milestones, on level ground and in the same vertical plane with the observer, are found to be 5° and 10° respectively. Find the height of the hill and the horizontal distance to the nearest milestone.

5. From a point distant 200 feet from the base of a tower the angles of elevation of the top of the tower and of a spire (which surmounts the tower) are found to be 35° 15′ and 42° 30′ respectively; find the heights of the tower and of the flagstaff.

6. AB is a line 1000 yards long; B is due north of A and from B a distant point P bears 70° east of north; at A it bears 41° 22′ east of north; find the distance from A to P.

7. A, B and C are three church towers; A and B are 3 miles apart and A bears N.W. from B; from C, A bears N. 35° E. and B bears N. 70° E.; find the distances of A and B from C.

8. From a lighthouse two ships A and B are seen in directions S.S.E. and E. 10° N. Also the distance between A and B is 2 miles and from A, B is seen in a direction N. 15° E. Find the distances of the two ships from the lighthouse.

9. P and Q are two stations 1000 yards apart on a straight stretch of sea shore, which bears East and West. At P a rock bears 42° West of South and at Q it bears 35° East of South. Shew that the distance of the rock from the shore is $1000\,\frac{\sin 48°\sin 55°}{\sin 77°}$ yards, and calculate this distance to the nearest yard.

[If R be the rock, then $\angle RPQ = 48°$ and $\angle RQP = 55°$, so that

$$\angle QRP = 180° - 48° - 55° = 77°.$$

If x be the length of the perpendicular from R upon PQ, then

$$\frac{x}{RP} = \sin 48°,$$

and
$$\frac{RP}{1000} = \frac{\sin Q}{\sin R} = \frac{\sin 55°}{\sin 77°}.$$

Hence, by multiplication,

$$\frac{x}{1000} = \frac{\sin 48°\sin 55°}{\sin 77°}].$$

10. A and B are two points, which are on the banks of a river and opposite to one another, and between them is the mast, PN, of a ship; the breadth of the river is 1000 feet, and the angular elevation of P at A is 14° 20′ and at B it is 8° 10′. What is the height of P above AB?

11. A is a station exactly 10 miles west of B. The bearing of a particular rock from A is 74° 19′ east of north, and its bearing from B is 26° 51′ west of north. How far is it north of the line AB?

12. A road 1890 yards long runs N.E. and S.W. From one end of it the base of a spire bears W. 18° 17′ S. From the other end of the road it bears N. 21° 39′ E. Find the distance of the spire from the road.

13. A tower PN stands on level ground. A base AB is measured at right angles to AN, the points A, B, and N being in the same horizontal plane, and the angles PAN and PBN are found to be α and β respectively. Prove that the height of the tower is

$$AB\frac{\sin\alpha\sin\beta}{\sqrt{\sin(\alpha-\beta)\sin(\alpha+\beta)}}.$$

If $AB=100$ feet, $\alpha=70°$, and $\beta=50°$, calculate the height.

14. A man, standing due south of a tower on a horizontal plane through its foot, finds the elevation of the top of the tower to be $54°\ 16'$; he goes east 100 yards and finds the elevation to be then $50°\ 8'$. Find the height of the tower.

15. A man in a balloon observes that the angle of depression of an object on the ground bearing due north is $33°$; the balloon drifts 3 miles due west and the angle of depression is now found to be $21°$. Find the height of the balloon.

16. A man observes that the angular elevation of the top of a monument is α; he then walks a distance a directly towards the monument up a slope whose inclination is β and then finds the elevation to be γ. Shew that the height of the top of the monument above the original place of observation is

$$a\sin\alpha\sin(\gamma-\beta)\operatorname{cosec}(\gamma-\alpha).$$

Obtain the numerical result when

$$a=100\text{ ft.},\ \alpha=23°\ 15',\ \beta=10°\ 10',\text{ and }\gamma=30°\ 30'.$$

Verify by drawing a figure to scale.

17. A castle and a monument stand on the same horizontal plane. The height of the castle is 140 feet, and the angles of depression of the top and bottom of the monument as seen from the top of the castle are $40°$ and $80°$ respectively. Find the height of the monument.

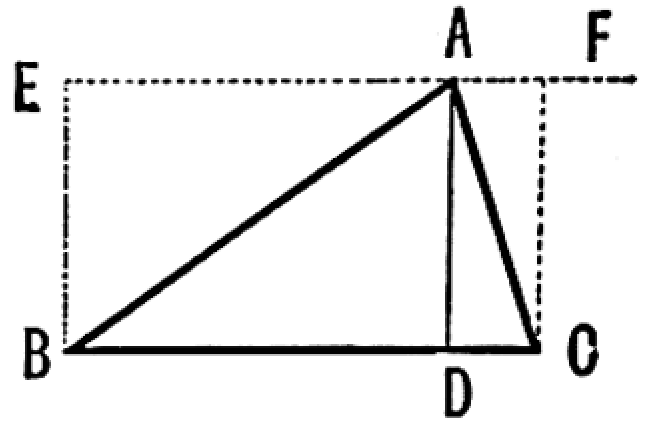
18. A man is walking up stream along the bank of a river which runs E. to W. He observes two trees, A and B, on the opposite bank; A is then bearing N. $29°\ 17'$ E. After walking 225 feet A is now bearing due N. and B is bearing N. $43°\ 13'$ E. Find the breadth of the river and the distance between the trees to the nearest foot; find also the bearing of B at the first place of observation to the nearest minute.

19. From the extremities of a horizontal base-line AB, whose length is 1000 feet, the bearings of the foot C of a tower are observed and it is found that $\angle CAB=56°\ 23'$, $\angle CBA=47°\ 15'$, and that the elevation of the tower from A is $9°\ 25'$; find the height of the tower.

CHAPTER XIII.

PROPERTIES OF A TRIANGLE.

133. Area of a given triangle. Let ABC be any triangle, and AD the perpendicular drawn from A upon the opposite side.



Through A draw EAF parallel to BC, and draw BE and CF perpendicular to it. By geometry, the area of the triangle ABC

$$=\tfrac{1}{2}\text{ rectangle } BF=\tfrac{1}{2}BC \,.\, CF=\tfrac{1}{2}a \,.\, AD.$$

But $AD=AB\sin B=c\sin B$.

The area of the triangle ABC therefore $=\frac{1}{2}ca\sin B$. This area is denoted by Δ.

Hence $\mathbf{\Delta=\tfrac{1}{2}ca\sin B=\tfrac{1}{2}ab\sin C=\tfrac{1}{2}bc\sin A}$...(1).

By Art. 105, we have $\sin A=\dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$,

so that $\Delta=\tfrac{1}{2}bc\sin A=\mathbf{\sqrt{s(s-a)(s-b)(s-c)}}$...(2).

This latter quantity is often called S.

Again, since $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C},$$

$$\therefore c=a\frac{\sin C}{\sin A}.$$

Hence, from (1),

$$\Delta=\frac{1}{2}ca\sin B=\frac{a^2}{2}\frac{\sin B\sin C}{\sin A},$$

and similarly

$$=\frac{b^2}{2}\frac{\sin C\sin A}{\sin B}=\frac{c^2}{2}\frac{\sin A\sin B}{\sin C}\ \ldots\ldots(3).$$

EXAMPLES. XXIX.

Find the area of the triangle ABC when

1. $a=13$, $b=14$, and $c=15$.

2. $a=18$, $b=24$, and $c=30$.

3. $a=25$, $b=52$, and $c=63$.

4. $a=125$, $b=123$, and $c=62$.

5. $a=15$, $b=36$, and $c=39$.

6. If $B=45°$, $C=60°$, and $a=2(\sqrt{3}+1)$ inches, prove that the area of the triangle is $6+2\sqrt{3}$ sq. inches.

7. The sides of a triangle are 119, 111, and 92 yards; prove that its area is 10 sq. yards less than an acre.

8. The sides of a triangular field are 242, 1212, and 1450 yards; prove that the area of the field is 6 acres.

Making use of the logarithm tables find the area of the triangles where

9. $a=5{\cdot}2$, $b=4{\cdot}5$, and $c=6{\cdot}7$ ins.

10. $a=487$, $b=648$, and $c=739$ feet.

11. $a=423$, $b=357$ feet, and $C=53°15'$.

12. $a=27$ feet, $B=65°25'$, and $C=73°20'$.

Find the area of a quadrilateral piece of ground $ABCD$ when

13. $AB = 240$ ft.; $BC = 400$ ft.; $CD = 180$ ft.; $DA = 420$ ft.; $AC = 480$ ft.

14. $AB = 50$ yds.; $BC = 120$ yds.; $CD = 70$ yds.; $DA = 100$ yds.; $\angle B = 90°$.

15. $AB = 120$ yds.; $BC = 90$ yds.; $AD = 150$ yds.; $AC = 150$ yds., and $\angle ADC = 50°$.

16. Shew that the area of a triangle $ABC = s^2 \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.

17. Find the other sides of the triangle in which the base is 6·3 ins., one base angle is 30°, and whose area is 7·875 sq. ins.

18. If $a = 28$ feet, $\sin C = \frac{3}{5}$ and the area be equal to that of a triangle whose sides are 45, 39, and 42 feet, find the other two sides.

19. A workman is told to make a triangular enclosure of sides 50, 41, and 21 yards respectively; having made the first side one yard too long, what length must he make the other two sides in order to enclose the prescribed area with the prescribed length of fencing?

20. Find, correct to ·0001 of an inch, the length of one of the equal sides of an isosceles triangle on a base of 14 inches having the same area as a triangle whose sides are 13·6, 15, and 15·4 inches.

21. The sides of a triangle are in A.P. and its area is $\frac{3}{5}$ths of an equilateral triangle of the same perimeter; prove that its sides are in the ratio 3 : 5 : 7, and find the greatest angle of the triangle.

134. On the circles connected with a given triangle.

The circle which passes through the angular points of a triangle ABC is called its circumscribing circle or, more briefly, its **circumcircle**. The centre of this circle is found by the construction of Art. 135. Its radius is always called R.

The circle which can be inscribed within the triangle so as to touch each of the sides is called its inscribed circle or, more briefly, its **incircle**. The centre of this circle is found by the construction of Art. 137. Its radius will be denoted by r.

The circle which touches the side BC and the two sides AB and AC produced is called the **escribed** circle opposite the angle A. Its radius will be denoted by r_1.

Similarly r_2 denotes the radius of the circle which touches the side CA and the two sides BC and BA produced. Also r_3 denotes the radius of the circle touching AB and the two sides CA and CB produced.

135. *To find the magnitude of R, the radius of the circumcircle of any triangle ABC.*

Bisect the two sides BC and CA in D and E respectively, and draw DO and EO perpendicular to BC and CA.

By geometry, O is the centre of the circumcircle. Join OB and OC.

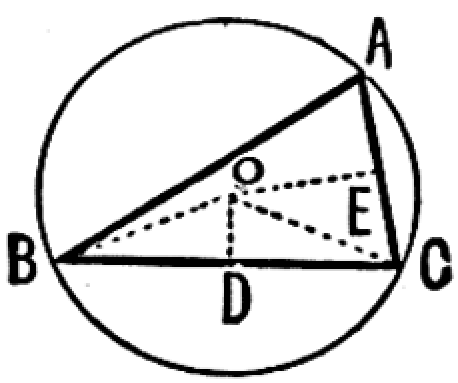


Fig. 1.

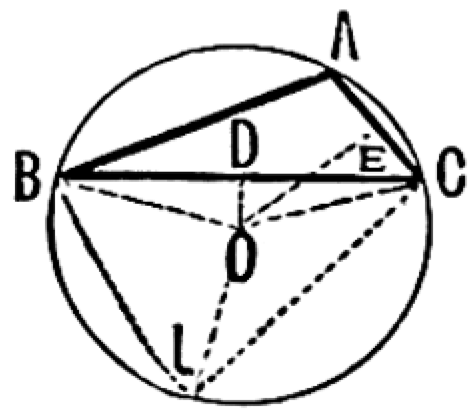


Fig. 2.

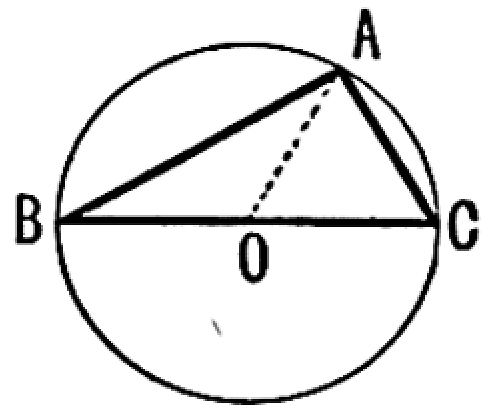


Fig. 3.

The point O may either lie within the triangle as in Fig. 1, or without it as in Fig. 2, or upon one of the sides as in Fig. 3.

Taking the first figure, the two triangles BOD and COD are equal in all respects, so that

$$\angle BOD = \angle COD,$$

$$\therefore \angle BOD = \tfrac{1}{2} \angle BOC = \angle BAC,$$
$$= A.$$

Also $\quad BD = BO \sin BOD.$

$$\therefore \frac{a}{2} = R \sin A.$$

If A be obtuse, as in Fig. 2, we have

$$\angle BOD = \tfrac{1}{2} \angle BOC = \angle BLC = 180^\circ - A,$$

so that, as before, $\sin BOD = \sin A$,

and $$R = \frac{a}{2 \sin A}.$$

If A be a right angle, as in Fig. 3, we have

$$R = OA = OC = \frac{a}{2}$$

$$= \frac{a}{2 \sin A}, \text{ since in this case } \sin A = 1.$$

The relation found above is therefore true for all triangles.

Hence, in all three cases, we have

$$\mathbf{R} = \frac{\mathbf{a}}{\mathbf{2 \sin A}} = \frac{\mathbf{b}}{\mathbf{2 \sin B}} = \frac{\mathbf{c}}{\mathbf{2 \sin C}} \quad \text{(Art. 99).}$$

136. In Art. 105 we have shewn that

$$\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)s-c)} = \frac{2S}{bc},$$

where S is the area of the triangle.

Substituting this value of $\sin A$ in (1), we have

$$\mathbf{R} = \frac{\mathbf{abc}}{\mathbf{4S}},$$

giving the radius of the circumcircle in terms of the sides.

137. *To find the value of r, the radius of the incircle of the triangle ABC.*

Bisect the two angles B and C by the two lines BI and CI meeting in I.

By geometry, I is the centre of the incircle. Join IA, and draw ID, IE, and IF perpendicular to the three sides.

Then $ID = IE = IF = r$.

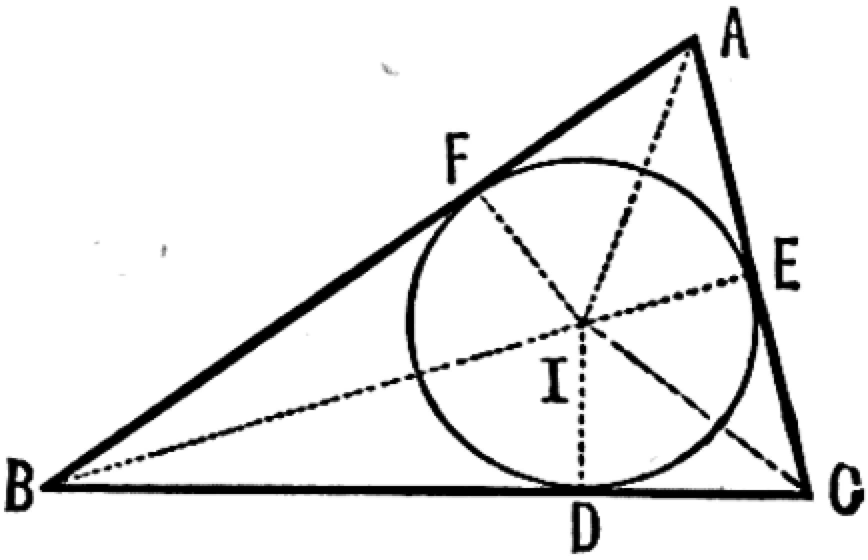
We have

$$\text{area of } \triangle\, IBC = \tfrac{1}{2} ID \,.\, BC = \tfrac{1}{2} r \,.\, a,$$

$$\text{area of } \triangle\, ICA = \tfrac{1}{2} IE \,.\, CA = \tfrac{1}{2} r \,.\, b,$$

and

$$\text{area of } \triangle\, IAB = \tfrac{1}{2} IF \,.\, AB = \tfrac{1}{2} r \,.\, c.$$



Hence, by addition, we have

$$\tfrac{1}{2} r \,.\, a + \tfrac{1}{2} r \,.\, b + \tfrac{1}{2} r \,.\, c = \text{sum of the areas of the triangles } IBC, ICA, \text{ and } IAB$$

$$= \text{area of the } \triangle\, ABC,$$

i.e.

$$r \frac{a+b+c}{2} = S,$$

so that

$$r \,.\, s = S.$$

$$\therefore \mathbf{r} = \frac{\mathbf{S}}{\mathbf{s}}.$$

138. Since the angles IBD and IDB are respectively equal to the angles IBF and IFB, the two triangles IDB and IFB are equal in all respects.

Hence $\quad BD = BF$, so that $2BD = BD + BF$.

So also $\quad AE = AF$, so that $2AE = AE + AF$,

and $\quad CE = CD$, so that $2CE = CE + CD$.

Hence, by addition, we have

$$2BD + 2AE + 2CE = (BD + CD) + (CE + AE) + (AF + FB),$$

i.e.

$$2BD + 2AC = BC + CA + AB.$$

$$\therefore\ 2BD + 2b = a + b + c = 2s.$$

Hence $$BD = s - b = BF;$$
so $$CE = s - c = CD,$$
and $$AF = s - a = AE.$$

Now $$\frac{ID}{BD} = \tan IBD = \tan\frac{B}{2}.$$

$$\therefore\ r = ID = BD\tan\frac{B}{2} = (s-b)\tan\frac{B}{2}.$$

So $$r = IE = CE\tan ICE = (s-c)\tan\frac{C}{2},$$

and also $$r = IF = FA\tan IAF = (s-a)\tan\frac{A}{2}.$$

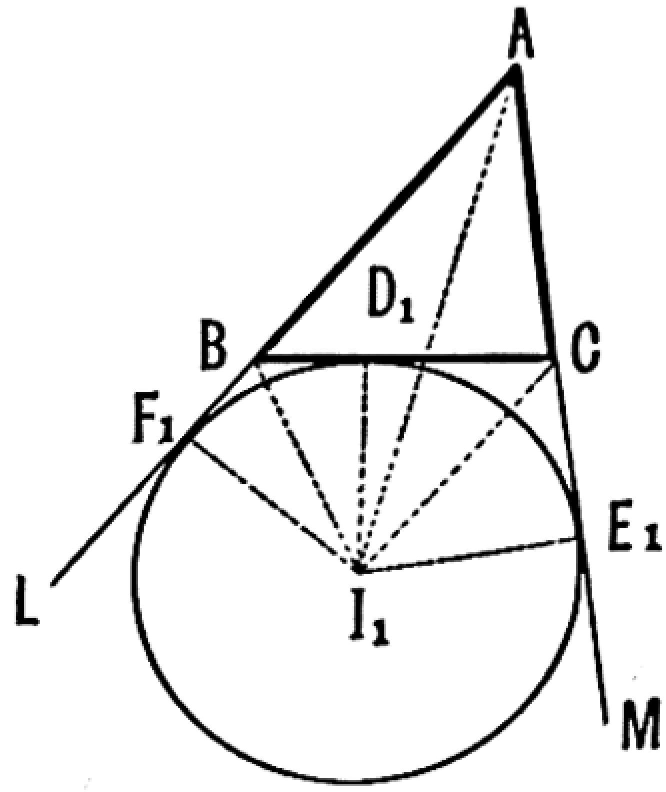
Hence

$$\mathbf{r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}.}$$

139. *To find the value of r_1, the radius of the escribed circle opposite the angle A of the triangle ABC.*

Produce AB and AC to L and M

Bisect the angles CBL and BCM by the lines BI_1 and CI_1, and let these lines meet in I_1.



Draw I_1D_1, I_1E_1, and I_1F_1 perpendicular to the three sides respectively.

The two triangles I_1D_1B and I_1F_1B are equal in all respects, so that $$I_1F_1 = I_1D_1.$$

Similarly $I_1E_1 = I_1D_1$.

The three perpendiculars I_1D_1, I_1E_1, and I_1F_1 being equal, the point I_1 is the centre of the required circle.

Now the area ABI_1C is equal to the sum of the triangles ABC and I_1BC; it is also equal to the sum of the triangles I_1BA and I_1CA.

Hence $\triangle ABC + \triangle I_1BC = \triangle I_1CA + \triangle I_1AB$.

$$\therefore\ S + \tfrac{1}{2}I_1D_1 . BC = \tfrac{1}{2}I_1E_1 . CA + \tfrac{1}{2}I_1F_1 . AB,$$

i.e.
$$S + \tfrac{1}{2}r_1 . a = \tfrac{1}{2}r_1 . b + \tfrac{1}{2}r_1 . c.$$

$$\therefore\ S = r_1\left[\frac{b+c-a}{2}\right] = r_1\left[\frac{b+c+a}{2} - a\right] = r_1(s-a).$$

$$\therefore\ \mathbf{r_1 = \frac{S}{s-a}}.$$

Similarly it can be shewn that

$$\mathbf{r_2 = \frac{S}{s-b}},\ \text{and}\ \mathbf{r_3 = \frac{S}{s-c}}.$$

140. Since AE_1 and AF_1 are tangents, we have as in Art. 138, $AE_1 = AF_1$.

Similarly, $BF_1 = BD_1$, and $CE_1 = CD_1$.

$$\therefore\ 2AE_1 = AE_1 + AF_1 = AB + BF_1 + AC + CE_1$$
$$= AB + BD_1 + AC + CD_1 = AB + BC + CA = 2s.$$
$$\therefore\ AE_1 = s = AF_1.$$
$$\therefore\ I_1E_1 = AE_1 \tan I_1AE_1,$$

i.e.
$$\mathbf{r_1 = s \tan \frac{A}{2}}.$$

Also, $BD_1 = BF_1 = AF_1 - AB = s - c$,

and $CD_1 = CE_1 = AE_1 - AC = s - b$.

$$\therefore\ r_1 = I_1F_1 = BF_1 \tan I_1BF_1 = BF_1 \tan \frac{180° - B}{2}$$
$$= (s-c)\tan\left(90° - \frac{B}{2}\right) = (s-c)\cot\frac{B}{2}.$$

So, similarly, $r_1 = I_1E_1 = (s-b)\cot\dfrac{C}{2}$.

EXAMPLES. XXX.

1. In a triangle whose sides are 18, 24, and 30 inches respectively, prove that the circumradius, the inradius, and the radii of the three escribed circles are respectively 15, 6, 12, 18, and 36 inches.

2. The sides of a triangle are 13, 14, and 15 feet; prove that

(1) $R = 8\frac{1}{8}$ ft., (2) $r = 4$ ft., (3) $r_1 = 10\frac{1}{2}$ ft.,

(4) $r_2 = 12$ ft., and (5) $r_3 = 14$ ft.

3. In a triangle ABC if $a=13$, $b=4$, and $\cos C=-\frac{5}{13}$, find R, r, r_1, r_2, and r_3.

4. In the ambiguous case of the solution of triangles prove that the circumcircles of the two triangles are equal.

5. In an equilateral triangle shew that the radii of the inscribed, circumscribed and escribed circles are as 1 : 2 : 3.

Prove that

6. $rr_1=bc\sin^2\frac{A}{2}=S\tan\frac{A}{2}$.

7. $r_2r_3=bc\cos^2\frac{A}{2}$.

8. $rr_1r_2r_3=S^2$.

9. $r=a\dfrac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$.

10. $r_1=a\dfrac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$.

11. $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

12. $r_1=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

13. $r_1r_2r_3=r^3\cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}$.

14. $r_2r_3+r_3r_1+r_1r_2=s^2$.

15. $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r}=0$.

16. $a(rr_1+r_2r_3)=b(rr_2+r_3r_1)=c(rr_3+r_1r_2)$.

17. $4sRr=abc$.

18. $S=2R^2\sin A\sin B\sin C$.

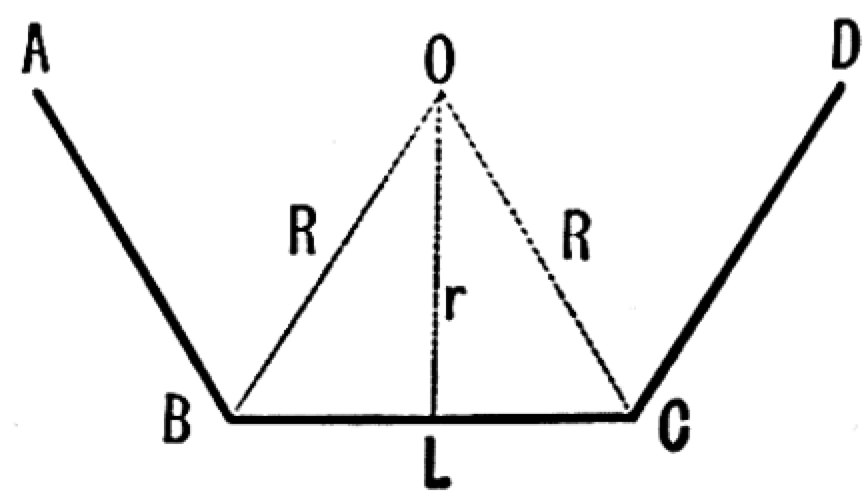
19. $\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{a^2+b^2+c^2}{S^2}$.

20. $r_1+r_2+r_3-r=4R$.

21. $\frac{1}{2Rr}=\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}$.

141. *Radii of the inscribed and circumscribing circles of a regular polygon.*

Let AB, BC, and CD be three successive sides of the polygon, and let n be the number of its sides.



Bisect the angles ABC and BCD by the lines BO and CO which meet in O, and draw OL perpendicular to BC.

It is easily seen that O is the centre of both

the incircle and the circumcircle of the polygon, and that BL equals LC.

Hence we have $OB = OC = R$, the radius of the circumcircle, and $OL = r$, the radius of the incircle.

The angle BOC is $\frac{1}{n}$th of the sum of all the angles subtended at O by the sides,

i.e.
$$\angle BOC = \frac{4 \text{ right angles}}{n} = \frac{360°}{n}.$$

Hence
$$\angle BOL = \tfrac{1}{2} \angle BOC = \frac{180°}{n}.$$

If a be a side of the polygon, we have

$$a = BC = 2BL = 2R \sin BOL = 2R \sin \frac{180°}{n}.$$

$$\therefore\ R = \frac{a}{2 \sin \frac{180°}{n}} = \frac{a}{2} \operatorname{cosec} \frac{180°}{n} \quad \ldots\ldots\ldots (1).$$

Again,
$$a = 2BL = 2OL \tan BOL = 2r \tan \frac{180°}{n}.$$

$$\therefore\ r = \frac{a}{2 \tan \frac{180°}{n}} = \frac{a}{2} \cot \frac{180°}{n} \quad \ldots\ldots\ldots\ldots (2).$$

142. Area of a Regular Polygon.

The area of the polygon is n times the area of the triangle BOC.

Hence the area of the polygon

$$= n \times \tfrac{1}{2} OL \,.\, BC = n \,.\, OL \,.\, BL = n \,.\, BL \cot LOB \,.\, BL$$
$$= n \,.\, \frac{a^2}{4} \cot \frac{180°}{n} \quad \ldots\ldots\ldots (1),$$

an expression for the area in terms of the side.

Also the area

$$= n \,.\, OL \,.\, BL = n \,.\, OL \,.\, OL \tan BOL = nr^2 \tan \frac{180°}{n} \ldots (2).$$

Again, the area

$$=n\,.\,OL\,.\,BL=n\,.\,OB\cos LOB\,.\,OB\sin LOB$$

$$=nR^2\cos\frac{180^\circ}{n}\sin\frac{180^\circ}{n}=\frac{n}{2}R^2\sin\frac{360^\circ}{n}\quad\text{.........(3).}$$

The formulae (2) and (3) give the area in terms of the radius of the inscribed and circumscribed circles.

143. **Ex.** *The length of each side of a regular dodecagon is* 20 *feet; find* (1) *the radius of its inscribed circle,* (2) *the radius of its circumscribing circle, and* (3) *its area.*

The angle subtended by a side at the centre of the polygon

$$=\frac{360^\circ}{12}=30^\circ.$$

Hence we have $\quad 10=r\tan 15^\circ=R\sin 15^\circ.$

$$\therefore\ r=10\cot 15^\circ$$

$$=\frac{10}{2-\sqrt{3}}\quad\text{(Art. 55)}$$

$$=10(2+\sqrt{3})=37\cdot32\ldots\text{ feet.}$$

Also $\quad R=\dfrac{10}{\sin 15^\circ}=10\times\dfrac{2\sqrt{2}}{\sqrt{3}-1}\quad$ (Art. 47)

$$=10\,.\,\sqrt{2}(\sqrt{3}+1)=10(\sqrt{6}+\sqrt{2})$$

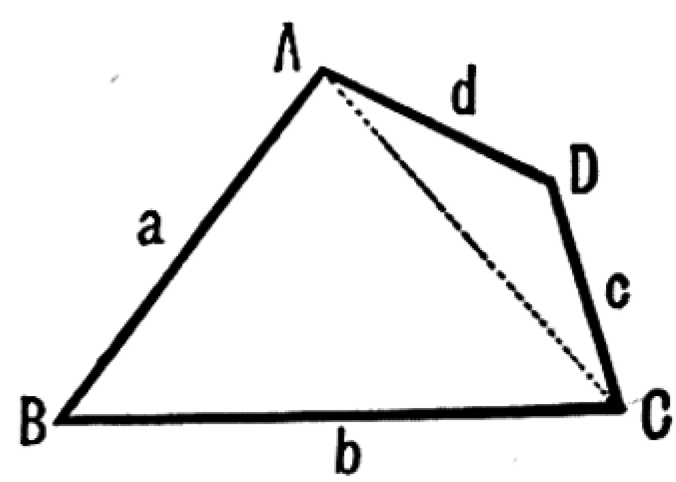
$$=10(2\cdot4495\ldots+1\cdot4142\ldots)=38\cdot637\ldots\text{ feet.}$$

Again, the area $\quad=12\times r\times 10$ square feet

$$=1200(2+\sqrt{3})=4478\cdot46\ldots\text{ square feet.}$$

144. **Area of a quadrilateral. If we have any quadrilateral we can express its area in terms of its sides and the sum of any two opposite angles.

For let the sum of the two angles B and D be denoted by $2a$, and denote the area of the quadrilateral by Δ.



Then

$$\Delta=\text{area of } ABC+\text{area of } ACD$$

$$=\tfrac{1}{2}ab\sin B+\tfrac{1}{2}cd\sin D,$$

so that

$$4\Delta=2ab\sin B+2cd\sin D\ldots(1).$$

Also

$$a^2+b^2-2ab\cos B=AC^2=c^2+d^2-2cd\cos D,$$

so that

$$a^2+b^2-c^2-d^2=2ab\cos B-2cd\cos D \quad\text{.............. (2).}$$

Squaring (1) and (2) and adding, we have

$$16\Delta^2+(a^2+b^2-c^2-d^2)^2=4a^2b^2+4c^2d^2$$
$$-8abcd(\cos B\cos D-\sin B\sin D)$$
$$=4a^2b^2+4c^2d^2-8abcd\cos(B+D)$$
$$=4a^2b^2+4c^2d^2-8abcd\cos 2\alpha$$
$$=4a^2b^2+4c^2d^2-8abcd(2\cos^2\alpha-1)$$
$$=4(ab+cd)^2-16abcd\cos^2\alpha,$$

so that

$$16\Delta^2=4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2-16abcd\cos^2\alpha \quad\text{...... (3).}$$

But

$$4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$$
$$=[2(ab+cd)+(a^2+b^2-c^2-d^2)][2(ab+cd)-(a^2+b^2-c^2-d^2)]$$
$$=[(a^2+2ab+b^2)-(c^2-2cd+d^2)][(c^2+2cd+d^2)-(a^2-2ab+b^2)]$$
$$=[(a+b)^2-(c-d)^2][(c+d)^2-(a-b)^2]$$
$$=(a+b+c-d)(a+b-c+d)[c+d+a-b][c+d-a+b].$$

Let $\qquad 2s=a+b+c+d,$

so that $\qquad a+b+c-d=(a+b+c+d)-2d=2(s-d),$

$$a+b-c+d=2(s-c), \text{ etc.}$$

Hence

$$4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2=16(s-a)(s-b)(s-c)(s-d),$$

so that (3) becomes

$$\Delta^2=(s-a)(s-b)(s-c)(s-d)-abcd\cos^2\alpha \quad\text{......... (4),}$$

giving the required area.

Cor. 1. If the quadrilateral be inscribable in a circle, so that $\angle B+\angle D=$ two right angles, then $2\alpha=180°$, i.e. $\alpha=90°$ and $\cos\alpha=0$.

In this case (4) gives

$$\Delta=\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

Cor. 2. If d be zero, the quadrilateral becomes a triangle, and the formula above becomes that of Art. 133.

EXAMPLES. XXXI.

1. Find, correct to $\cdot 01$ of an inch, the length of the perimeter of a regular decagon which surrounds a circle of radius one foot. [$\tan 18^\circ = \cdot 32492$.]

2. Find to 3 places of decimals the length of the side of a regular polygon of 12 sides which is circumscribed to a circle of unit radius.

3. Find the area of (1) a pentagon, (2) a hexagon, (3) an octagon, (4) a decagon and (5) a dodecagon, each being a regular figure of side 1 foot. [$\cot 18^\circ = 3\cdot 07768\ldots$; $\cot 36^\circ = 1\cdot 37638\ldots$.]

4. Find the difference between the areas of a regular octagon and a regular hexagon if the perimeter of each be 24 feet.

5. A square, whose side is 2 feet, has its corners cut away so as to form a regular octagon; find its area.

6. Compare the areas and perimeters of octagons which are respectively inscribed in and circumscribed to a given circle, and shew that the areas of the inscribed hexagon and octagon are as $\sqrt{27}$ to $\sqrt{32}$.

7. Prove that the radius of the circle described about a regular pentagon is nearly $\frac{17}{20}$ths of the side of the pentagon.

8. If an equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are as $2 : 3$.

9. If a regular pentagon and a regular decagon have the same perimeter, prove that their areas are as $2 : \sqrt{5}$.

10. The area of a regular polygon of n sides inscribed in a circle is to that of the same number of sides circumscribing the same circle as 3 is to 4. Find the value of n.

CHAPTER XIV.

CENTESIMAL AND CIRCULAR MEASURE OF ANGLES.

145. It has been pointed out in Chapter I that the Sexagesimal System is not a very convenient one on account of the inconvenient multipliers 60 and 90. Hence another system of measurement called the **Centesimal**, or French, system has been proposed. In this system the right angle is divided into 100 equal parts, called **Grades**; each grade is subdivided into 100 **Minutes**, and each minute into 100 **Seconds**.

The symbols 1^g, $1^`$, and $1^{``}$ are used to denote a Grade, a Minute, and a Second respectively.

Thus 100 Seconds ($100^{``}$) make One Minute ($1^`$),

100 Minutes ($100^`$) „ „ Grade (1^g),

100 Grades (100^g) „ „ Right angle.

This system would be much more convenient to use than the ordinary Sexagesimal System.

As a preliminary, however, to its practical adoption, a large number of tables would have to be recalculated. For this reason the system has in practice never been used.

146. *To convert Sexagesimal into Centesimal Measure, and vice versa.*

Since a right angle is equal to 90° and also to 100^g, we have

$$90^\circ = 100^g.$$

$$\therefore\ 1^\circ = \frac{10^g}{9},\ \text{and}\ 1^g = \frac{9^\circ}{10}.$$

Hence, to change degrees into grades, add on one-ninth; to change grades into degrees, subtract one-tenth.

Ex. $$36^\circ = \left(36 + \frac{1}{9} \times 36\right)^g = 40^g,$$

and $$64^g = \left(64 - \frac{1}{10} \times 64\right)^\circ = (64 - 6{\cdot}4)^\circ = 57{\cdot}6^\circ.$$

If the angle do not contain an integral number of degrees, we may reduce it to a fraction of a degree and then change to grades.

In practice it is generally found more convenient to reduce any angle to a fraction of a right angle. The method will be seen in the following examples:

Ex. 1. *Reduce* $63^\circ 14' 51''$ *to Centesimal Measure.*

We have $$51'' = \frac{17'}{20} = {\cdot}85',$$

and $$14'51'' = 14{\cdot}85' = \frac{14{\cdot}85^\circ}{60} = {\cdot}2475^\circ,$$

$$\therefore\ 63^\circ 14' 51'' = 63{\cdot}2475^\circ = \frac{63{\cdot}2475}{90}\ \text{rt. angle}$$

$$= {\cdot}70275\ \text{rt. angle}$$

$$= 70{\cdot}275^g = 70^g\, 27{\cdot}5^` = 70^g\, 27^`\, 50^{``}.$$

Ex. 2. *Reduce* $94^g\, 23^`\, 87^{``}$ *to Sexagesimal Measure.*

$$\begin{array}{rl} 94^g\, 23^`\, 87^{``} = & {\cdot}942387 \text{ right angle} \\ & \underline{\qquad\quad 90} \\ & 84{\cdot}81483 \text{ degrees} \\ & \underline{\qquad\quad 60} \\ & 48{\cdot}8898 \text{ minutes} \\ & \underline{\qquad\quad 60} \\ & 53{\cdot}3880 \text{ seconds.} \end{array}$$

$$\therefore\ 94^g\, 23^`\, 87^{``} = 84^\circ\, 48'\, 53{\cdot}388''.$$

EXAMPLES. XXXII.

Express in grades, minutes, and seconds the angles

1. $30°$. 2. $81°$. 3. $138°30'$.

4. $35°47'15''$. 5. $235°12'36''$.

6. $475°13'48''$.

Express in terms of right angles, and also in degrees, minutes, and seconds the angles

7. 120^g. 8. $45^g35`24``$.  9. $39^g45`36``$.

10. $255^g8`9``$.  11. $759^g0`5``$.

Mark the position of the revolving line when it has traced out the following angles:

12. 150^g. 13. 420^g. 14. 875^g.

15. The number of degrees in one acute angle of a right-angled triangle is equal to the number of grades in the other; express both the angles in degrees.

16. Show that the number of Sexagesimal minutes in any angle is to the number of Centesimal minutes in the same angle as 27 : 50.

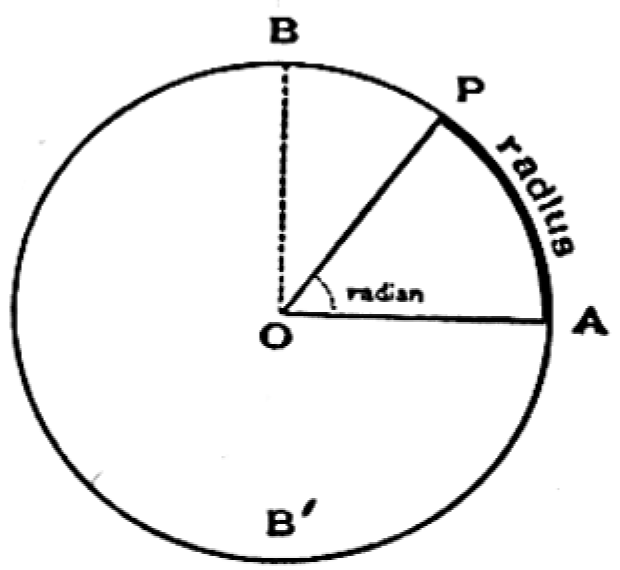
17. Divide $44°8'$ into two parts such that the number of Sexagesimal seconds in one part may be equal to the number of Centesimal seconds in the other.

Circular Measure.

147. A third system of measurement of angles has been devised, and it is this system which is used in all the higher branches of Mathematics.

The unit used is obtained thus;

Take *any* circle $APBB'$, whose centre is O, and from any point A measure off an arc AP whose length is equal to the radius of the circle. Join OA and OP.



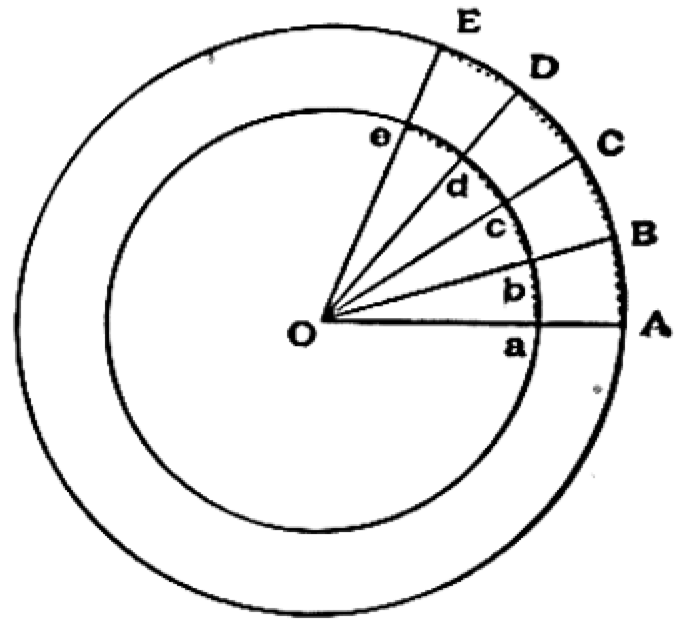
The angle AOP is the angle which is taken as the unit of circular measurement, *i.e.* it is the angle in terms of which in this system we measure all others.

This angle is called **A Radian** and is often denoted by 1^c.

It is clearly essential to the proper choice of a unit that it should be a *constant* quantity; hence we must shew that the Radian is a constant angle. This we shall do in the following articles.

148. Theorem. *The length of the circumference of a circle always bears a constant ratio to its diameter.*

Take any two circles whose common centre is O. In the large circle inscribe a regular polygon of n sides, $ABCD\ldots$



Let OA, OB, OC,... meet the smaller circle in the points a, b, c, d,... and join ab, bc, cd,...

Then, by geometry, $abcd\ldots$ is a regular polygon of n sides inscribed in the smaller circle.

Since $Oa = Ob$, and $OA = OB$, the lines ab and AB must be parallel, and hence

$$\frac{AB}{ab} = \frac{OA}{Oa}.$$

Also the polygon $ABCD\ldots$ being regular, its perimeter, *i.e.* the sum of its sides, is equal to $n\,.\,AB$. Similarly for the inner polygon.

Hence we have

$$\frac{\text{Perimeter of the outer polygon}}{\text{Perimeter of the inner polygon}} = \frac{n\,.\,AB}{n\,.\,ab} = \frac{AB}{ab} = \frac{OA}{Oa} \quad \ldots\ldots\ldots(1).$$

This relation exists whatever be the number of sides in the polygons.

Let then the number of sides be indefinitely increased (*i.e.* let n become inconceivably great) so that finally the perimeter of the outer polygon will be the same as the circumference of the outer circle, and the perimeter of the inner polygon the same as the circumference of the inner circle.

The relation (1) will then become

$$\frac{\text{Circumference of outer circle}}{\text{Circumference of inner circle}} = \frac{OA}{Oa}$$

$$= \frac{\text{Radius of outer circle}}{\text{Radius of inner circle}}.$$

Hence $\dfrac{\text{Circumference of outer circle}}{\text{Radius of outer circle}}$

$$= \frac{\text{Circumference of inner circle}}{\text{Radius of inner circle}}.$$

Since there was no restriction whatever as to the sizes of the two circles, it follows that the quantity

$$\frac{\text{Circumference of a circle}}{\text{Radius of the circle}}$$

is **the same for all circles.**

Hence the ratio of the circumference of a circle to its radius, and therefore also to its diameter, is a constant quantity.

149. In the previous article we have shown that the ratio $\dfrac{\text{Circumference}}{\text{Diameter}}$ is the same for all circles. The value of this constant ratio is always denoted by the Greek letter π (pronounced Pi), so that π is a number.

Hence $\dfrac{\text{Circumference}}{\text{Diameter}} = \text{the constant number } \pi.$

We have therefore the following theorem; **The circumference of a circle is always equal to π times its diameter or 2π times its radius.**

150. Unfortunately the value of π is not a whole number, nor can it be expressed in the form of a vulgar fraction, and hence not in the form of a decimal fraction, terminating or recurring.

The number π is an incommensurable magnitude, *i.e.* a magnitude whose value cannot be exactly expressed as the ratio of two whole numbers.

Its value, correct to 8 places of decimals, is

$$3{\cdot}14159265\ldots.$$

The fraction $\frac{22}{7}$ gives the value of π correctly for the first two decimal places; for $\frac{22}{7} = 3{\cdot}14285\ldots.$

The fraction $\frac{355}{113}$ is a more accurate value of π, being correct to 6 places of decimals; for $\frac{355}{113} = 3{\cdot}14159203\ldots.$

[N.B. The fraction $\frac{355}{113}$ may be remembered thus; write down the first three odd numbers repeating each twice, thus 113355; divide the number thus obtained into two equal portions and let the first part be divided into the second, thus 113)355(.

The quotient is the value of π to 6 places of decimals.]

To sum up. **An approximate value of π, correct to 2 places of decimals, is the fraction $\frac{22}{7}$; a more accurate value is 3·14159**....

By division, we can shew that

$$\frac{1}{\pi} = {\cdot}318309886 2\ldots.$$

151. **Ex. 1.** *The diameter of a bicycle wheel is 28 inches; through what distance does its centre move during one revolution of the wheel?*

The radius r is here 14 inches.

The circumference therefore $= 2 . \pi . 14 = 28\pi$ inches.

If we take $\pi = \frac{22}{7}$, the circumference $= 28 \times \frac{22}{7}$ inches $= 7$ ft. 4 inches approximately.

If we give π the more accurate value $3{\cdot}14159265\ldots$, the circumference

$$= 28 \times 3{\cdot}14159265\ldots \text{ inches} = 7 \text{ ft. } 3{\cdot}96459\ldots \text{ inches.}$$

Ex. 2. *What must be the radius of a circular running path, round which an athlete must run 5 times in order to describe one mile?*

The circumference must be $\frac{1}{5} \times 1760$, *i.e.* 352 yards.

Hence, if r be the radius of the path in yards, we have $2\pi r = 352$,

i.e. $$r = \frac{176}{\pi} \text{ yards.}$$

Taking $\pi = \frac{22}{7}$, we have $r = \frac{176 \times 7}{22} = 56$ yards nearly.

Taking the more accurate value $\frac{1}{\pi} = \cdot 31831$, we have

$$r = 176 \times \cdot 31831 = 56 \cdot 02256 \text{ yards.}$$

EXAMPLES. XXXIII.

1. If the radius of the earth be 4000 miles, what is the length of its circumference?

2. The wheel of a railway carriage is 3 feet in diameter and makes 3 revolutions in a second; how fast is the train going?

3. A mill sail whose length is 18 feet makes 10 revolutions per minute. What distance does its end travel in an hour?

4. The diameter of a halfpenny is an inch; what is the length of a piece of string which would just surround its curved edge?

5. Assuming that the earth describes in one year a circle, of 92500000 miles radius, whose centre is the sun, how many miles does the earth travel in a year?

6. The minute hand of a clock is 3 ft. 9 ins. long; through what distance does its end move in 20 minutes?

7. How many revolutions does a bicycle wheel, of diameter 28 inches, make in a mile?

8. The radius of a carriage wheel is 1 ft. 9 ins., and in $\frac{1}{9}$th of a second it turns through $80°$ about its centre, which is fixed; how many miles does a point on the rim of the wheel travel in one hour?

152. Theorem. *The radian is a constant angle.*

Take the figure of Art. 147. Let the arc AB be a quadrant of the circle, *i.e.* one quarter of the circumference.

By Art. 149, the length of AB is therefore $\frac{\pi r}{2}$, where r is the radius of the circle.

By geometry, we know that angles at the centre of any circle are to one another as the arcs on which they stand.

Hence $$\frac{\angle AOP}{\angle AOB}=\frac{\text{arc } AP}{\text{arc } AB}=\frac{r}{\frac{\pi}{2}r}=\frac{2}{\pi},$$

i.e. $$\angle AOP=\frac{2}{\pi}.\angle AOB.$$

But we defined the angle AOP to be a Radian.

Hence a Radian $=\frac{2}{\pi}.\ \angle AOB$

$=\frac{2}{\pi}$ of a right angle.

Since a right angle is a constant angle, and since we have shewn (Art. 149) that π is a constant quantity, it follows that a Radian is a constant angle, and is therefore the same whatever be the circle from which it is derived.

153. Magnitude of a Radian.

By the previous article, a Radian

$$=\frac{2}{\pi}\times \text{a right angle}=\frac{180^\circ}{\pi}$$
$$=180^\circ\times\cdot 3183098862\ldots=57\cdot 2957795^\circ\ldots$$
$$=57^\circ\ 17'\ 44\cdot 8'' \text{ nearly.}$$

154. Since a Radian $=\frac{2}{\pi}$ of a right angle,

therefore a right angle $=\frac{\pi}{2}$ radians,

so that $180^\circ = 2$ right angles $=\pi$ radians,

and $360^\circ = 4$ right angles $=2\pi$ radians.

Hence, when the revolving line has made a complete revolution, it has described an angle equal to 2π radians; when it has made three complete revolutions, it has described an angle of 6π radians; when it has made n revolutions, it has described an angle of $2n\pi$ radians.

155. In practice the symbol "c" is generally omitted, and instead of "an angle π^c" we find written "an angle π."

The student must notice this point carefully. If the unit, in terms of which the angle is measured, be not mentioned, he must mentally supply the word "radians." Otherwise he will easily fall into the mistake of supposing that π stands for $180°$. It is true that π radians (π^c) is the same as $180°$, but π itself is a number, and a number only.

156. *To convert circular measure into sexagesimal measure or centesimal measure and vice versa.*

The student should remember the relations

$$\text{Two right angles} = 180° = 200^g = \pi \text{ radians}.$$

The conversion is then merely Arithmetic.

Ex. (1) $\cdot 45\pi^c = \cdot 45 \times 180° = 81° = 90^g$.

(2) $3^c = \frac{3}{\pi} \times \pi^c = \frac{3}{\pi} \times 180° = \frac{3}{\pi} \times 200^g$.

(3) $40° 15' 36'' = 40° 15\tfrac{3}{5}' = 40\cdot 26°$

$$= 40\cdot 26 \times \frac{\pi^c}{180} = \cdot 223\dot{6}\pi \text{ radians}.$$

(4) $40^g 15^` 36^{``} = 40\cdot 1536^g = 40\cdot 1536 \times \frac{\pi}{200}$ radians

$$= \cdot 200768\pi \text{ radians}.$$

157. Ex. 1. *The angles of a triangle are in* A.P. *and the number of grades in the least is to the number of radians in the greatest as* $40 : \pi$; *find the angles in degrees.*

Let the angles be $(x-y)°$, $x°$, and $(x+y)°$.

Since the sum of the three angles of a triangle is $180°$, we have

$$180 = x - y + x + x + y = 3x,$$

so that $$x = 60.$$

The required angles are therefore

$$(60-y)°,\ 60°, \text{ and } (60+y)°.$$

Now $$(60-y)° = \frac{10}{9} \times (60-y)^g,$$

and $$(60+y)° = \frac{\pi}{180} \times (60+y) \text{ radians}.$$

Hence $$\frac{10}{9}(60-y) : \frac{\pi}{180}(60+y) :: 40 : \pi,$$

$$\therefore \frac{200}{\pi}\frac{60-y}{60+y}=\frac{40}{\pi},$$

i.e. $$5(60-y)=60+y,$$

i.e. $$y=40.$$

The angles are therefore 20°, 60°, and 100°.

Ex. 2. *Express in the 3 systems of angular measurement the magnitude of the angle of a regular decagon.*

By geometry we know that all the interior angles of any rectilinear figure together with four right angles are equal to twice as many right angles as the figure has sides.

Let the angle of a regular decagon contain x right angles, so that all the angles are together equal to $10x$ right angles.

The corollary therefore states that

$$10x+4=20,$$

so that $$x=\frac{8}{5} \text{ right angles.}$$

But one right angle

$$=90^\circ=100^g=\frac{\pi}{2} \text{ radians.}$$

Hence the required angle

$$=144^\circ=160^g=\frac{4\pi}{5} \text{ radians.}$$

EXAMPLES. XXXIV.

Express in degrees, minutes, and seconds the angles,

1. $\frac{\pi^c}{3}$. **2.** $\frac{4\pi^c}{3}$. **3.** $10\pi^c$. **4.** 1^c. **5.** 8^c.

Express in grades, minutes, and seconds the angles,

6. $\frac{4\pi^c}{5}$. **7.** $\frac{7\pi^c}{6}$. **8.** $10\pi^c$.

Express in radians the following angles:

9. 60°. **10.** 110° 30′. **11.** 175° 45′. **12.** 47° 25′ 36″.

13. 395°. **14.** 60^g. **15.** $110^g 30^`$. **16.** $345^g 25^` 36^{``}$.

17. The difference between the two acute angles of a right-angled triangle is $\frac{2}{5}\pi$ radians; express the angles in degrees.

18. The circular measure of two angles of a triangle are respectively $\frac{1}{2}$ and $\frac{1}{3}$; what is the number of degrees in the third angle?

19. The angles of a triangle are in A.P. and the number of degrees in the least is to the number of radians in the greatest as 60 to π; find the angles in degrees.

20. The angles of a triangle are in A.P. and the number of radians in the least angle is to the number of degrees in the mean angle as 1 : 120. Find the angles in radians.

21. Find the magnitude, in radians and degrees, of the interior angle of (1) a regular pentagon, (2) a regular heptagon, (3) a regular octagon, (4) a regular duodecagon, and (5) a regular polygon of 17 sides.

22. The hour, minute and second hands of a watch are respectively ·6, ·75, and ·3 inches long; compare the rates at which their extremities move.

23. One angle of a quadrilateral is 60 degrees, a second is 50 grades and a third is $\frac{3\pi}{4}$ radians; express all four angles in degrees.

24. The exterior angle of a regular polygon is one-sixth of the interior angle; find the number of its sides and express its angle in radians.

25. The angle in one regular polygon is to that in another as 3 : 2; also the number of sides in the first is twice that in the second; how many sides have the polygons?

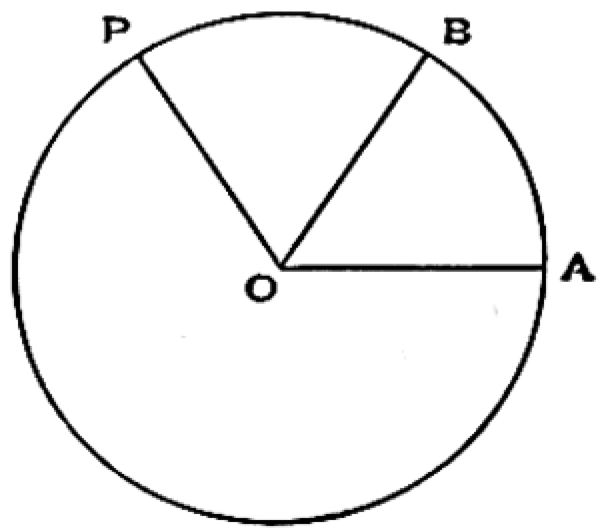
26. The angles of a quadrilateral are in A.P. and the greatest is double the least; express the least angle in radians.

27. Find in radians, degrees, and grades the angle between the hour-hand and the minute-hand of a clock at (1) half-past three, (2) twenty minutes to six, (3) a quarter past eleven.

28. Find the times (1) between four and five o'clock when the angle between the minute-hand and the hour-hand is 78°, (2) between seven and eight o'clock when this angle is 54°.

158. Theorem. *The number of radians in any angle whatever is equal to a fraction, whose numerator is the arc which the angle subtends at the centre of any circle, and whose denominator is the radius of that circle.*

Let AOP be the angle which has been described by a line starting from OA and revolving into the position OP.



With centre O and any radius describe a circle cutting OA and OP in the points A and P.

Let the angle AOB be a radian, so that the arc AB is equal to the radius OA.

By geometry, we have

$$\frac{\angle AOP}{\text{A Radian}} = \frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } AB} = \frac{\text{arc } AP}{\text{Radius}},$$

so that
$$\angle AOP = \frac{\text{arc } AP}{\text{Radius}} \text{ of a Radian.}$$

Hence the theorem is proved.

159. Ex. 1. *Find the angle subtended at the centre of a circle of radius 3 feet by an arc of length 1 foot.*

The number of radians in the angle $= \dfrac{\text{arc}}{\text{radius}} = \dfrac{1}{3}$.

Hence the angle

$$= \frac{1}{3} \text{ radian} = \frac{1}{3} \cdot \frac{2}{\pi} \text{ right angle} = \frac{2}{3\pi} \times 90^\circ = \frac{60^\circ}{\pi} = 19\tfrac{1}{11}^\circ,$$

π being taken equal to $\dfrac{22}{7}$.

Ex. 2. *In a circle of 5 feet radius what is the length of the arc which subtends an angle of 33° 15′ at the centre?*

If x feet be the required length, we have

$$\frac{x}{5} = \text{number of radians in } 33^\circ 15'$$
$$= \frac{33\frac{1}{4}}{180}\pi \quad \text{(Art. 156)}$$
$$= \frac{133}{720}\pi.$$

$$\therefore\ x=\frac{133}{144}\pi \text{ feet}=\frac{133}{144}\times\frac{22}{7} \text{ feet nearly}$$
$$=2\frac{65}{72} \text{ feet nearly.}$$

Ex. 3. *Assuming the average distance of the earth from the sun to be* 92500000 *miles, and the angle subtended by the sun at the eye of a person on the earth to be* 32′, *find the sun's diameter.*

Let D be the diameter of the sun in miles.

The angle subtended by the sun being very small, its diameter is very approximately equal to a small arc of a circle whose centre is the eye of the observer. Also the sun subtends an angle of 32′ at the centre of this circle.

Hence, by Art. 158, we have

$$\frac{D}{92500000}=\text{the number of radians in } 32'$$
$$=\text{the number of radians in } \frac{8^\circ}{15}$$
$$=\frac{8}{15}\times\frac{\pi}{180}=\frac{2\pi}{675}.$$
$$\therefore\ D=\frac{185000000}{675}\pi \text{ miles}$$
$$=\frac{185000000}{675}\times\frac{22}{7} \text{ miles approximately}$$
$$=\text{about } 862000 \text{ miles.}$$

Ex. 4. *Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of* 5′ *at his eye, find what is the height of the letters that he can read at a distance* (1) *of* 12 *feet, and* (2) *of a quarter of a mile.*

Let x be the required height in feet.

In the first case, x is very nearly equal to the arc of a circle, of radius 12 feet, which subtends an angle of 5′ at its centre.

Hence
$$\frac{x}{12}=\text{number of radians in } 5'$$
$$=\frac{1}{12}\times\frac{\pi}{180}.$$
$$\therefore\ x=\frac{\pi}{180} \text{ feet}=\frac{1}{180}\times\frac{22}{7} \text{ feet nearly}$$
$$=\frac{1}{15}\times\frac{22}{7} \text{ inches}=\text{about } \frac{1}{5} \text{ inch.}$$

In the second case, the height y is given by

$$\frac{y}{440 \times 3} = \text{number of radians in } 5'$$
$$= \frac{1}{12} \times \frac{\pi}{180},$$

so that
$$y = \frac{11}{18}\pi = \frac{11}{18} \times \frac{22}{7} \text{ feet nearly}$$
$$= \text{about } 23 \text{ inches}$$

EXAMPLES. XXXV.

[*Take* $\pi = 3{\cdot}14159$, $\frac{1}{\pi} = {\cdot}31831$, *and* $\log \pi = {\cdot}49715$.]

1. Find the number of degrees subtended at the centre of a circle by an arc whose length is ·357 times the radius.

2. Express in radians and degrees the angle subtended at the centre of a circle by an arc whose length is 15 feet, the radius of the circle being 25 feet.

3. The value of the divisions on the outer rim of a graduated circle is $5'$ and the distance between successive graduations is ·1 inch. Find the radius of the circle.

4. The diameter of a graduated circle is 6 feet and the graduations on its rim are $5'$ apart; find the distance from one graduation to another.

5. Find the radius of a globe which is such that the distance between two places on the same meridian whose latitude differs by $1^\circ 10'$ may be half-an-inch.

6. Taking the earth as a sphere of radius 4000 miles, find the difference in latitude of two places, one of which is 100 miles north of the other.

7. Assuming the earth to be a sphere and the distance between two parallels of latitude, which subtends an angle of 1° at the earth's centre, to be $69\frac{1}{9}$ miles, find the radius of the earth.

8. The radius of a certain circle is 3 feet; find approximately the length of an arc of this circle, if the length of the chord of the arc be 3 feet also.

9. What is the ratio of the radii of two circles at the centre of which two arcs of the same length subtend angles of 60° and 75°?

10. If an arc, of length 10 feet, on a circle of 8 feet diameter subtend at the centre an angle of $143° 14' 22''$; find the value of π to 4 places of decimals.

11. If the circumference of a circle be divided into 5 parts which are in A.P., and if the greatest part be 6 times the least, find in radians the magnitudes of the angles that the parts subtend at the centre of the circle.

12. At what distance does a man, whose height is 6 feet, subtend an angle of $10'$?

13. Find the length which at a distance of one mile will subtend an angle of $1'$ at the eye.

14. Find approximately the distance at which a globe, $5\frac{1}{2}$ inches in diameter, will subtend an angle of $6'$.

15. Find approximately the distance of a tower whose height is 51 feet and which subtends at the eye an angle of $5\frac{5}{11}'$.

16. A church spire, whose height is known to be 100 feet, subtends an angle of $9'$ at the eye; find approximately its distance.

17. Find approximately in minutes the inclination to the horizon of an incline which rises $3\frac{1}{2}$ feet in 210 yards.

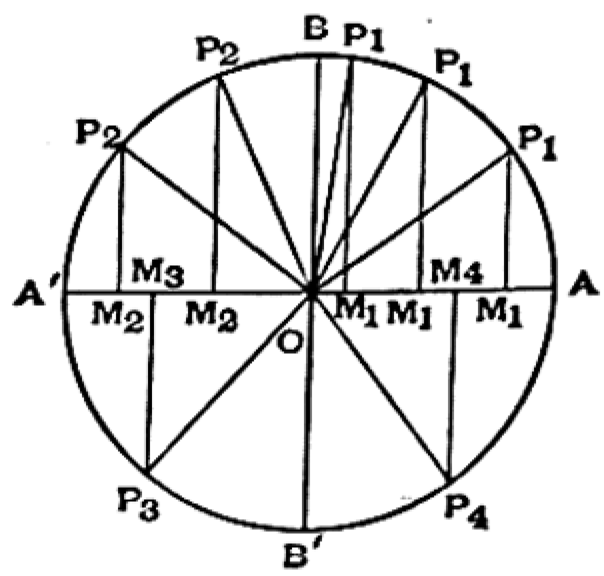
18. The radius of the earth being taken to be 3960 miles, and the distance of the moon from the earth being 60 times the radius of the earth, find approximately the radius of the moon which subtends at the earth an angle of $16'$.

CHAPTER XV.

TRIGONOMETRICAL FUNCTIONS OF ANGLES OF ANY MAGNITUDE. GRAPHS OF TRIGONOMETRICAL FUNCTIONS. CONNECTED ANGLES.

160. *To trace the changes in the sign and magnitude of the trigonometrical ratios of an angle, as the angle increases from* 0° *to* 360°.

Let the revolving line OP be of constant length a.



When it coincides with OA, the length OM_1 is equal to a and, when it coincides with OB, the point M_1 coincides with O and OM_1 vanishes. Also, as the revolving line turns from OA to OB, the distance OM_1 decreases from a to zero.

Whilst the revolving line is in the second quadrant and is revolving from OB to OA', the distance OM_2 is negative and increases numerically from 0 to a [*i.e.* it *decreases algebraically* from 0 to $-a$].

In the third quadrant, the distance OM_3 *increases* algebraically from $-a$ to 0, and, in the fourth quadrant, the distance OM_4 increases from 0 to a.

In the first quadrant, the length M_1P_1 increases from 0 to a; in the second quadrant, M_2P_2 decreases from a to 0; in the third quadrant, M_3P_3 decreases algebraically from 0 to $-a$; whilst in the fourth quadrant M_4P_4 increases algebraically from $-a$ to 0.

161. Sine. In the first quadrant, as the angle increases from 0 to 90°, the sine, *i.e.* $\frac{M_1P_1}{a}$, increases from $\frac{0}{a}$ to $\frac{a}{a}$, *i.e.* from 0 to 1.

In the second quadrant, as the angle increases from 90° to 180°, the sine decreases from $\frac{a}{a}$ to $\frac{0}{a}$, *i.e.* from 1 to 0.

In the third quadrant, as the angle increases from 180° to 270°, the sine *decreases* from $\frac{0}{a}$ to $\frac{-a}{a}$, *i.e.* from 0 to -1.

In the fourth quadrant, as the angle increases from 270° to 360°, the sine *increases* from $\frac{-a}{a}$ to $\frac{0}{a}$, *i.e.* from -1 to 0.

162. Cosine. In the first quadrant the cosine, which is equal to $\frac{OM}{a}$, decreases from $\frac{a}{a}$ to $\frac{0}{a}$, *i.e.* from 1 to 0.

In the second quadrant, it decreases from $\frac{0}{a}$ to $\frac{-a}{a}$, *i.e.* from 0 to -1.

In the third quadrant, it increases from $\frac{-a}{a}$ to $\frac{0}{a}$, *i.e.* from -1 to 0.

In the fourth quadrant, it increases from $\frac{0}{a}$ to $\frac{a}{a}$, *i.e.* from 0 to 1.

163. Tangent. In the first quadrant, M_1P_1 increases from 0 to a and OM_1 decreases from a to 0, so that $\frac{M_1P_1}{OM_1}$ continually increases (for its numerator continually increases and its denominator continually decreases).

When OP_1 coincides with OA, the tangent is 0; when the revolving line has turned through an angle which is slightly less than a right angle, so that OP_1 nearly

coincides with OB, then M_1P_1 is very nearly equal to a and OM_1 is very small. The ratio $\frac{M_1P_1}{OM_1}$ is therefore very large, and the nearer OP_1 gets to OB the larger does the ratio become, so that, by taking the revolving line near enough to OB, we can make the tangent as large as we please. This is expressed by saying that, when the angle is equal to 90°, its tangent is infinite.

The symbol ∞ is used to denote an infinitely great quantity.

Hence in the first quadrant the tangent increases from 0 to ∞.

In the second quadrant, when the revolving line has described an angle AOP_2 slightly greater than a right angle, M_2P_2 is very nearly equal to a and OM_2 is very small and negative, so that the corresponding tangent is very large and negative.

Also, as the revolving line turns from OB to OA', M_2P_2 decreases from a to 0 and OM_2 is negative and decreases from 0 to $-a$, so that when the revolving line coincides with OA' the tangent is zero.

Hence, in the second quadrant, the tangent increases from $-\infty$ to 0.

In the third quadrant, both M_3P_3 and OM_3 are negative, and hence their ratio is positive. Also, when the revolving line coincides with OB', the tangent is infinite.

Hence, in the third quadrant, the tangent increases from 0 to ∞.

In the fourth quadrant, M_4P_4 is negative and OM_4 is positive, so that their ratio is negative. Also, as the revolving line passes through OB' the tangent changes from $+\infty$ to $-\infty$ [just as in passing through OB].

Hence, in the fourth quadrant, the tangent increases from $-\infty$ to 0.

164. Cotangent. When the revolving line coincides with OA, M_1P_1 is very small and OM_1 is very nearly

equal to a, so that the cotangent, *i.e.* the ratio $\frac{OM_1}{M_1P_1}$, is infinite to start with. Also, as the revolving line rotates from OA to OB, the quantity M_1P_1 increases from 0 to a and OM_1 decreases from a to 0.

Hence, in the first quadrant, the cotangent decreases from ∞ to 0.

In the second quadrant, M_2P_2 is positive and OM_2 negative, so that the cotangent decreases from 0 to $\frac{-a}{0}$, *i.e.* from 0 to $-\infty$.

In the third quadrant, it is positive and decreases from ∞ to 0 [for as the revolving line crosses OA' the cotangent changes from $-\infty$ to ∞].

In the fourth quadrant, it is negative and decreases from 0 to $-\infty$.

165. Secant. When the revolving line coincides with OA the value of OM_1 is a, so that the value of the secant is then unity.

As the revolving line turns from OA to OB, OM_1 decreases from a to 0, and when the revolving line coincides with OB the value of the secant is $\frac{a}{0}$, *i.e.* ∞.

Hence, in the first quadrant, the secant increases from 1 to ∞.

In the second quadrant, OM_2 is negative and decreases from 0 to $-a$. Hence, in this quadrant, the secant increases from $-\infty$ to -1 [for as the revolving line crosses OB the quantity OM_1 changes sign and therefore the secant changes from $+\infty$ to $-\infty$].

In the third quadrant, OM_3 is always negative and increases from $-a$ to 0; therefore the secant decreases from -1 to $-\infty$. In the fourth quadrant, OM_4 is always positive and increases from 0 to a. Hence, in this quadrant, the secant decreases from ∞ to $+1$.

166. Cosecant. The change in the cosecant may be traced in a similar manner to that in the secant.

In the first quadrant, it decreases from ∞ to $+1$.

In the second quadrant, it increases from $+1$ to $+\infty$.

In the third quadrant, it increases from $-\infty$ to -1.

In the fourth quadrant, it decreases from -1 to $-\infty$.

167. The foregoing results are collected in the annexed table.

B

In the second quadrant, the			In the first quadrant, the		
sine	decreases from	1 to 0	sine	increases from	0 to 1
cosine	decreases from	0 to -1	cosine	decreases from	1 to 0
tangent	increases from	$-\infty$ to 0	tangent	increases from	0 to ∞
cotangent	decreases from	0 to $-\infty$	cotangent	decreases from	∞ to 0
secant	increases from	$-\infty$ to -1	secant	increases from	1 to ∞
cosecant	increases from	1 to ∞	cosecant	decreases from	∞ to 1

A′ O A

In the third quadrant, the			In the fourth quadrant, the		
sine	decreases from	0 to -1	sine	increases from	-1 to 0
cosine	increases from	-1 to 0	cosine	increases from	0 to 1
tangent	increases from	0 to ∞	tangent	increases from	$-\infty$ to 0
cotangent	decreases from	∞ to 0	cotangent	decreases from	0 to $-\infty$
secant	decreases from	-1 to $-\infty$	secant	decreases from	∞ to 1
cosecant	increases from	$-\infty$ to -1	cosecant	decreases from	-1 to $-\infty$

B′

168. Periods of the trigonometrical functions.

As an angle increases from 0 to 2π radians, *i.e.* whilst the revolving line makes a complete revolution, its sine first increases from 0 to 1, then decreases from 1 to -1, and finally increases from -1 to 0, and thus the sine goes through all its changes, returning to its original value.

Similarly, as the angle increases from 2π radians to 4π radians, the sine goes through the same series of changes.

Also, the sines of any two angles which differ by four right angles, *i.e.* 2π radians, are the same.

This is expressed by saying that the **period of the sine is 2π.**

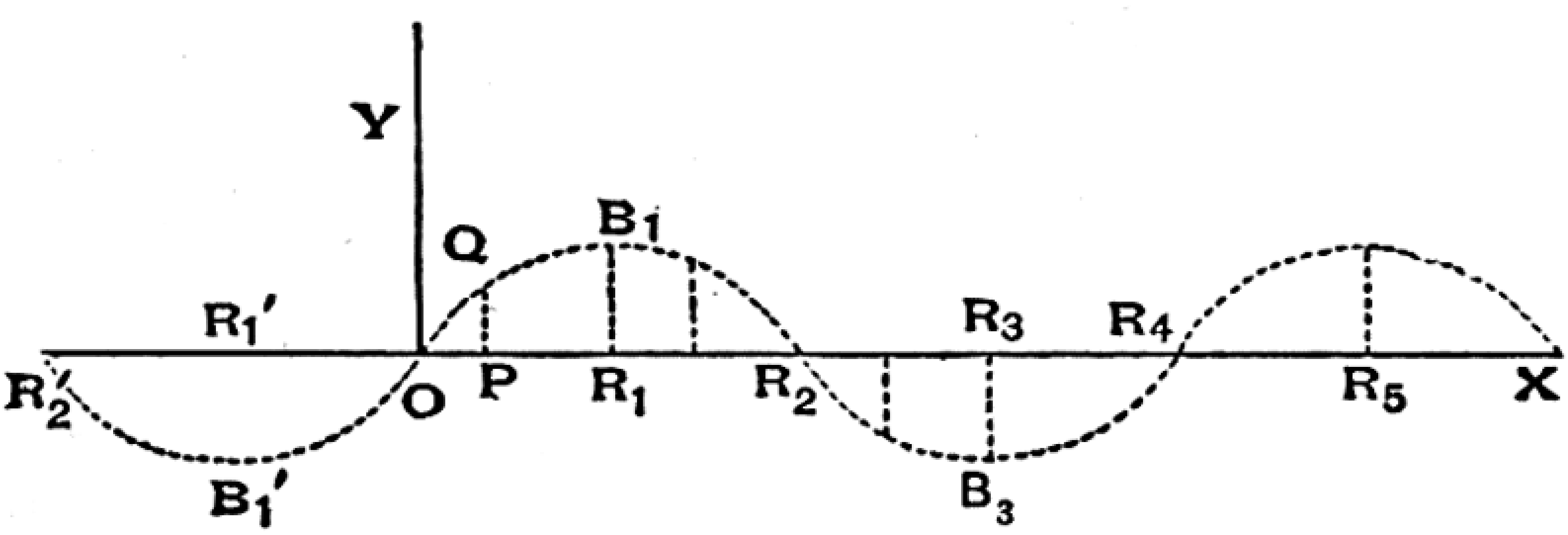
Similarly, the cosine, secant, and cosecant go through all their changes as the angle increases by 2π.

The tangent, however, goes through all its changes as the angle increases from 0 to π radians, *i.e.* whilst the revolving line turns through two right angles. Similarly for the cotangent.

The period of the sine, cosine, secant and cosecant is therefore 2π radians; the period of the tangent and cotangent is π radians.

Since the values of the trigonometrical functions repeat over and over again as the angle increases, they are called **periodic functions.**

169. The variations in the values of the trigonometrical ratios may be graphically represented to the eye by means of curves constructed in the following manner.



Sine-Graph.

Let OX and OY be two straight lines at right angles and let the magnitudes of **angles** be represented by **lengths** measured along OX.

Let R_1, R_2, R_3,... be points such that the distances OR_1, R_1R_2, R_2R_3,... are equal. If then the distance OR_1 represent a right angle, the distances OR_2, OR_3, OR_4,... must represent two, three, four,... right angles.

Also, if P be *any* point on the line OX, then OP represents an angle which bears the same ratio to a right angle that OP bears to OR_1.

[For example, if OP be equal to $\frac{1}{3}$ OR_1, then OP would represent one-third of a right angle; if P bisected R_3R_4, then OP would represent $3\frac{1}{2}$ right angles.]

In a similar manner, negative angles are represented by distances OR_1', OR_2',... measured from O in a negative direction.

At each point P erect a perpendicular PQ to represent the sine of the angle which is represented by OP; if the sine be positive, the perpendicular is to be drawn parallel to OY in the positive direction; if the sine be negative, the line is to be drawn in the negative direction.

[For example, since OR_1 represents a right angle, the sine of which is 1, we erect a perpendicular R_1B_1 equal to one unit of length; since OR_2 represents an angle equal to two right angles, the sine of which is zero, we erect a perpendicular of length zero; since OR_3 represents three right angles, the sine of which is -1, we erect a perpendicular equal to -1, *i.e.* we draw R_3B_3 downward and equal to a unit of length; if OP were equal to one-third of OR_1, it would represent $\frac{1}{3}$ of a right angle, *i.e.* 30°, the sine of which is $\frac{1}{2}$, and so we should erect a perpendicular PQ equal to one-half the unit of length.]

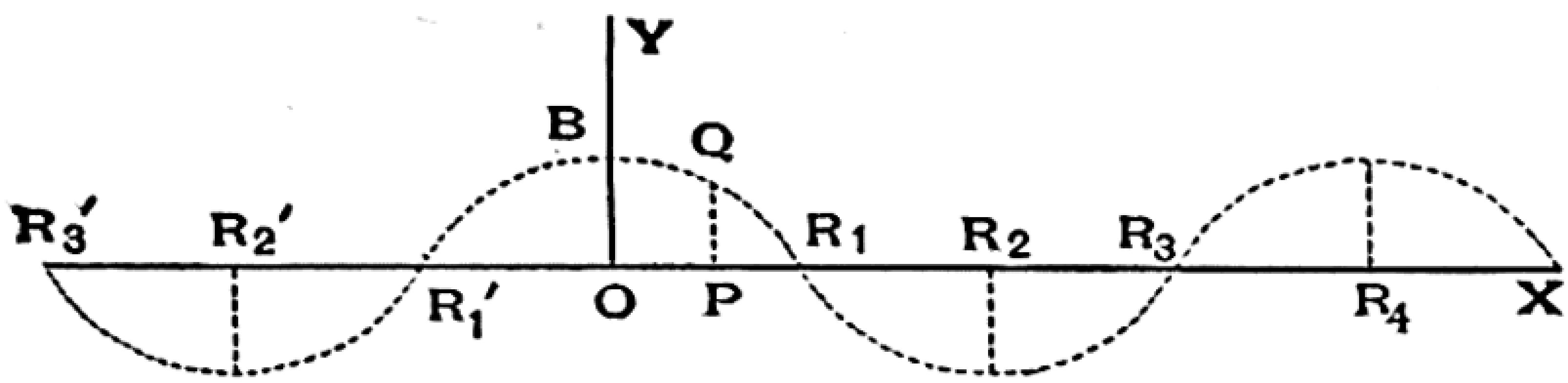
The ends of all these lines, thus drawn, would be found to lie on a curve similar to the one drawn above.

It would be found that the curve consisted of portions, similar to $OB_1R_2B_3R_4$, placed side by side. This corresponds to the fact that each time the angle increases by 2π, the sine repeats the same value.

In the above figure for lines parallel to OY unity is represented by ·25 inch, and, for distances measured along OX, one right angle is represented by ·4 inch.

170. Cosine-Graph.

The Cosine-Graph is obtained in the same manner as the Sine-Graph, except that in this case the perpendicular PQ represents the cosine of the angle represented by OP.

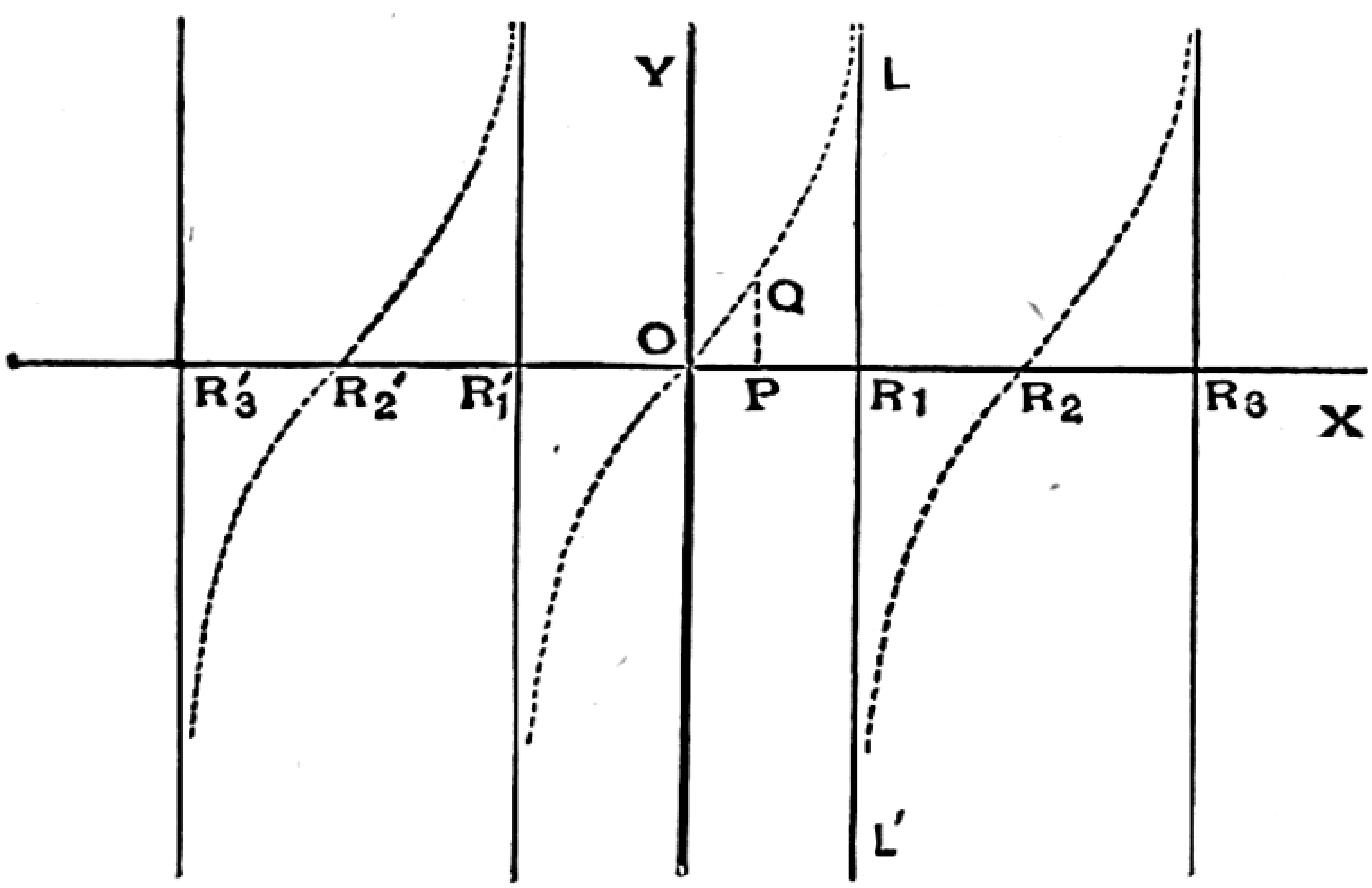


The curve obtained is the same as that of Art. 169 if in that curve we move O to R_1 and let OY be drawn along R_1B_1.

171. Tangent-Graph.

In this case, since the tangent of a right angle is infinite and since OR_1 represents a right angle, the perpendicular drawn at R_1 must be of infinite length and the dotted curve will only meet the line R_1L at an infinite distance.

Since the tangent of an angle slightly greater than a right angle is negative and almost infinitely great, the



dotted curve immediately beyond LR_1L' commences at an infinite distance on the negative side, *i.e.* below, OX.

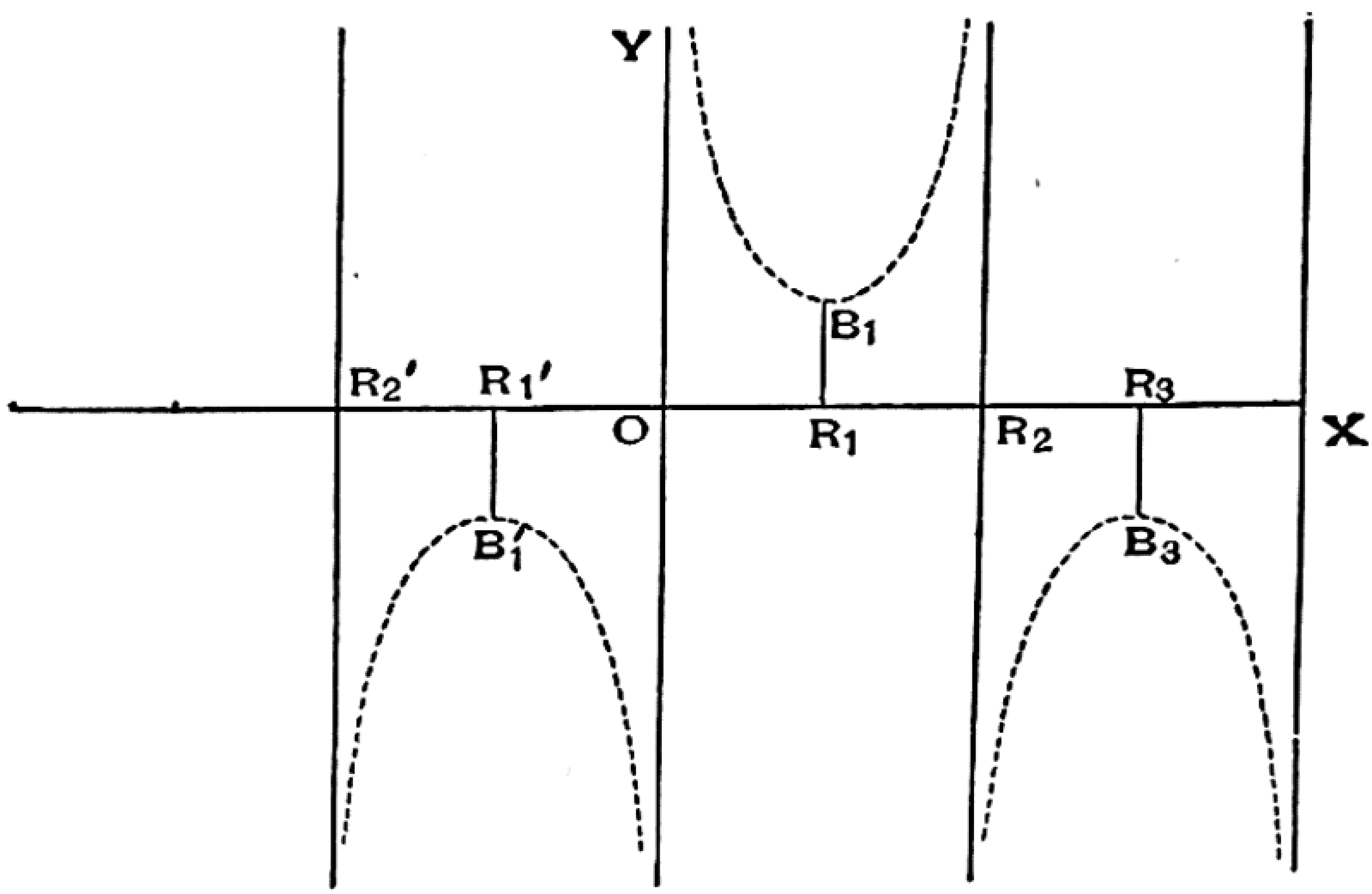
The Tangent-Graph will clearly consist of an infinite number of similar but disconnected portions, all ranged

parallel to one another. Such a curve is called a Discontinuous Curve. Both the Sine-Graph and the Cosine-Graph are, on the other hand, Continuous Curves.

172. Cotangent-Graph. If the curve to represent the cotangent be drawn in a similar manner, it will be found to meet OY at an infinite distance above O; it will pass through the point R_1 and touch the vertical line through R_2 at an infinite distance on the negative side of OX. Just beyond R_2 it will start at an infinite distance above R_2, and proceed as before.

The curve is therefore discontinuous and will consist of an infinite number of portions all ranged side by side.

173. Cosecant-Graph.



When the angle is zero, the sine is zero, and the cosecant is therefore infinite.

Hence the curve meets OY at infinity.

When the angle is a right angle, the cosecant is unity, and hence R_1B_1 is equal to the unit of length.

When the angle is equal to two right angles its cosecant is infinity, so that the curve meets the perpendicular through R_2 at an infinite distance.

Again, as the angle increases from slightly less to slightly greater than two right angles, the cosecant changes from $+\infty$ to $-\infty$.

Hence just beyond R_2 the curve commences at an infinite distance on the negative side of, *i.e.* below, OX.

174. Secant-Graph. If, similarly, the Secant-Graph be traced it will be found to be the same as the Cosecant-Graph would be if in it we moved OY to R_1B_1.

175. Ex. 1. *To trace the changes in the expression* $\sin x + \cos x$ *as x increases from* 0 *to* 2π.

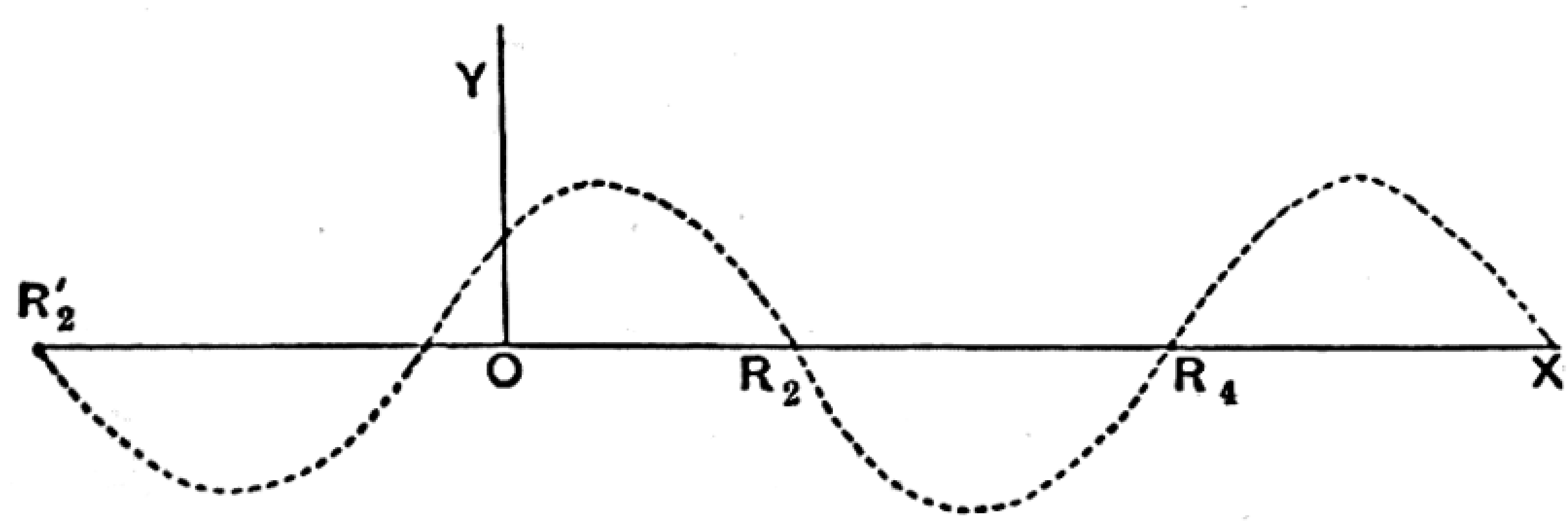
We have $\sin x + \cos x = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right]$

$$= \sqrt{2}\left[\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right] = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right).$$

We thus have the following table of values:

x	0	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	2π
$x+\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{9\pi}{4}$
$\sin\left(x+\frac{\pi}{4}\right)$	$\frac{1}{\sqrt{2}}$	1	0	-1	0	$\frac{1}{\sqrt{2}}$
$\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$	1	$\sqrt{2}$	0	$-\sqrt{2}$	0	1

As in Art. 169, the graph is drawn in the following figure.



Ex. 2. *To trace the changes in the sign and magnitude of* $a\cos\theta + b\sin\theta$, *and to find the greatest value of the expression.*

We have

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right].$$

Let α be the smallest positive angle such that

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \text{ and } \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

The expression therefore

$$= \sqrt{a^2+b^2}\,[\cos\theta\cos\alpha + \sin\theta\sin\alpha] = \sqrt{a^2+b^2}\cos(\theta - \alpha).$$

As θ changes from α to $2\pi + \alpha$, the angle $\theta - \alpha$ changes from 0 to 2π, and hence the changes in the sign and magnitude of the expression are easily obtained.

Since the greatest value of the quantity $\cos(\theta - \alpha)$ is unity, *i.e.* when θ equals α, the greatest value of the expression is $\sqrt{a^2+b^2}$.

Also the value of θ which gives this greatest value is such that its cosine is $\dfrac{a}{\sqrt{a^2+b^2}}$.

EXAMPLES. XXXVI.

As θ increases from 0 to 2π, trace the changes in the sign and magnitude of the following expressions, and plot their graphs.

1. $\sin\theta - \cos\theta$.
2. $\sin\theta + \sqrt{3}\cos\theta$.

[N.B. $\sin\theta + \sqrt{3}\cos\theta = 2\left[\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right] = 2\sin\left(\theta + \frac{\pi}{3}\right)$.]

3. $\sin\theta - \sqrt{3}\cos\theta$.
4. $\cos^2\theta - \sin^2\theta$.
5. $\sin\theta\cos\theta$.
6. $\tan 2\theta$.
7. $\sin 3\theta$.
8. $\tan 3\theta$.
9. $\sec 3\theta$.
10. $\operatorname{cosec} 4\theta$.
11. $\dfrac{\sin\theta + \sin 2\theta}{\cos\theta + \cos 2\theta}$.

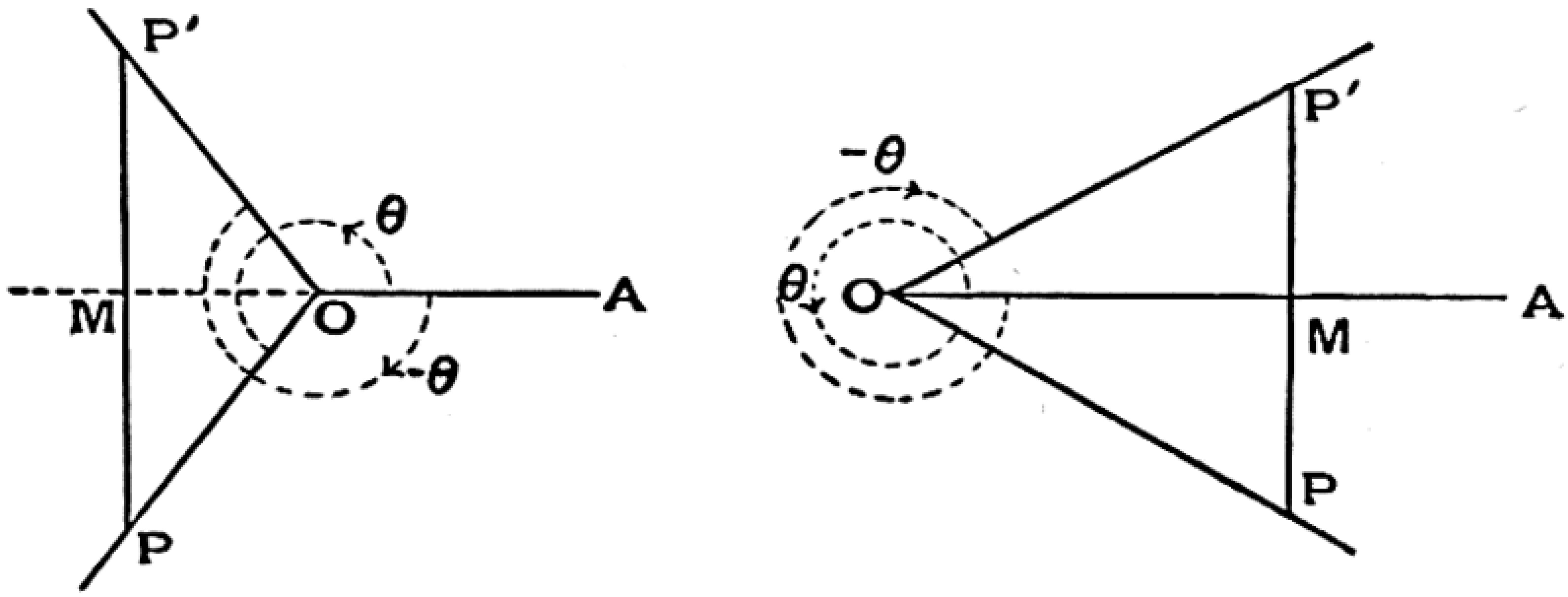
Find the greatest value of the following expressions, and state the corresponding values of θ;

12. $3\cos\theta + 4\sin\theta$.
13. $\cos\theta - \sin\theta$.
14. $2\cos\theta - \sin\theta$.

176. In Arts. 38—42 we have expressed the trigonometrical ratios of the angles $-\theta$, $90° - \theta$, $90° + \theta$, and $180° - \theta$ in terms of those of θ, for the cases in which θ is less than two right angles. We shall now show that the same relations hold for all values of θ. It will be only

necessary to draw additional figures, corresponding to the cases when the revolving line OP is in the third and fourth quadrants, and it will be seen that the proofs are exactly the same as those of these articles.

177. *To find the trigonometrical ratios of an angle* $(-\theta)$ *in terms of those of* θ, *for all values of* θ.



Let the revolving line, starting from OA, revolve through any angle θ and stop in the position OP.

Draw PM perpendicular to OA (or OA produced) and produce it to P', so that the lengths of PM and MP' are equal.

In the geometrical triangles MOP and MOP', we have the two sides OM and MP equal to the two OM and MP' and the included angles OMP and OMP' are right angles.

Hence, the magnitudes of the angles MOP and MOP' are the same, and OP is equal to OP'.

In each of the figures, the magnitudes of the angle AOP (measured ↺) and of the angle AOP' (measured ↻) are the same.

Hence the angle AOP' (measured ↻) is denoted by $-\theta$.

Also MP and MP' are equal in magnitude but are opposite in sign. (Art. 34.) We have therefore

$$\sin(-\theta) = \frac{MP'}{OP'} = \frac{-MP}{OP} = -\sin\theta,$$

$$\cos(-\theta) = \frac{OM}{OP'} = \frac{OM}{OP} = \cos\theta.$$

So for the other trigonometrical functions as in Art. 38.

178. *To find the trigonometrical ratios of the angle* $(90° - \theta)$ *in terms of those of* θ *for all values of* θ.

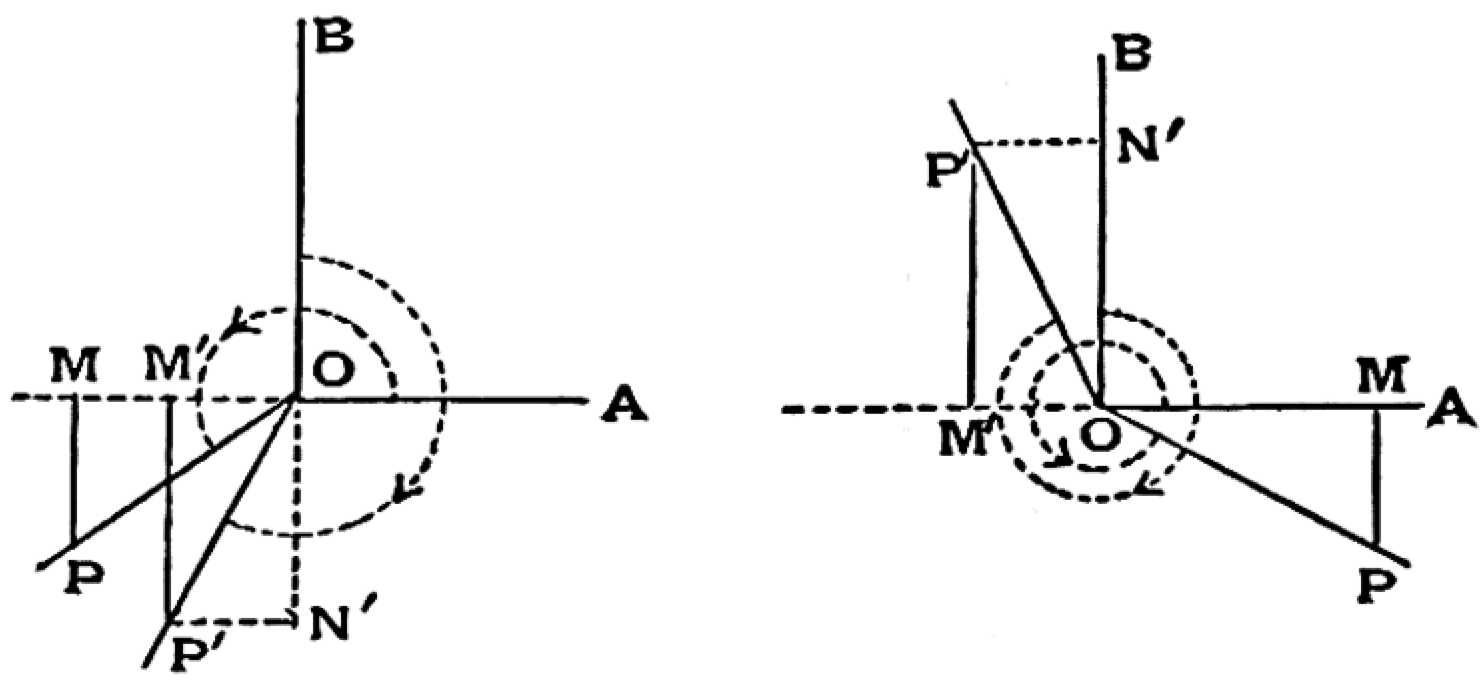
Let the revolving line, starting from OA, trace out any angle AOP denoted by θ.

To obtain the angle $90° - \theta$, let the revolving line rotate to B and then rotate from B in the opposite direction through the angle θ, and let the position of the revolving line be then OP'.

The angle AOP' is then $90° - \theta$.

Take OP' equal to OP, and draw $P'M'$ and PM perpendicular to OA, produced if necessary. Also draw $P'N'$ perpendicular to OB, produced if necessary.

In each figure, the angles AOP and BOP' are numerically equal, by construction.



Hence, in each figure,

$$\angle MOP = \angle N'OP' = \angle OP'M',$$

since ON' and $M'P'$ are parallel.

[For, in the second figure,

$$\angle MOP = 4 \text{ rt. } \angle^s - \angle AOP = 4 \text{ rt. } \angle^s - \angle P'OB = \angle N'OP'.]$$

Hence the triangles MOP and $M'P'O$ are equal in all respects, and therefore $OM = M'P'$ numerically,

and $$OM' = MP \text{ numerically.}$$

Also, in each figure, OM and $M'P'$ are of the same sign, and so also are MP and OM',

$$i.e.\ OM = +M'P', \text{ and } OM' = +MP.$$

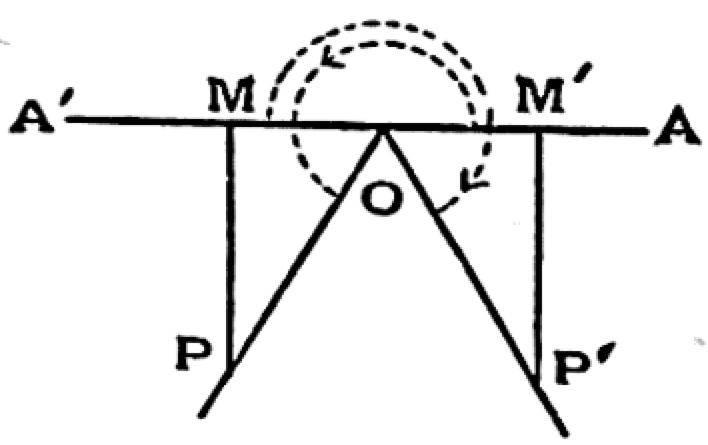
Hence

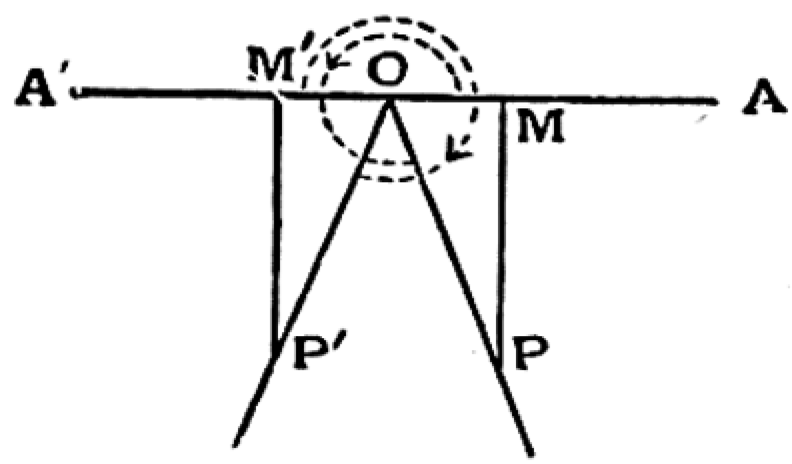
$$\sin(90^\circ - \theta) = \sin AOP' = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\theta,$$

$$\cos(90^\circ - \theta) = \cos AOP' = \frac{OM'}{OP'} = \frac{MP}{OP} = \sin\theta.$$

So for the other trigonometrical functions as in Art. 39.

179. *To find the values of the trigonometrical ratios of the angle* $(180^\circ - \theta)$ *in terms of those of the angle* θ, *for all values of* θ.





Let the revolving line start from OA and describe any angle AOP $(= \theta)$.

To obtain the angle $180^\circ - \theta$, let the revolving line start from OA and, after revolving through two right angles (*i.e.* into the position OA'), then revolve back through an angle θ into the position OP'. The angle $A'OP'$ is then equal in magnitude but opposite in sign to the angle AOP.

Also the angle AOP' is $180^\circ - \theta$.

Take OP' equal to OP, and draw $P'M'$ and PM perpendicular to AOA'.

The angles MOP and $M'OP'$ are equal, and hence the triangles MOP and $M'OP'$ are equal in all respects.

Hence OM and OM' are equal in magnitude, and so also are MP and $M'P'$.

In each figure, OM and OM' are drawn in opposite directions, whilst MP and $M'P'$ are drawn in the same direction, so that

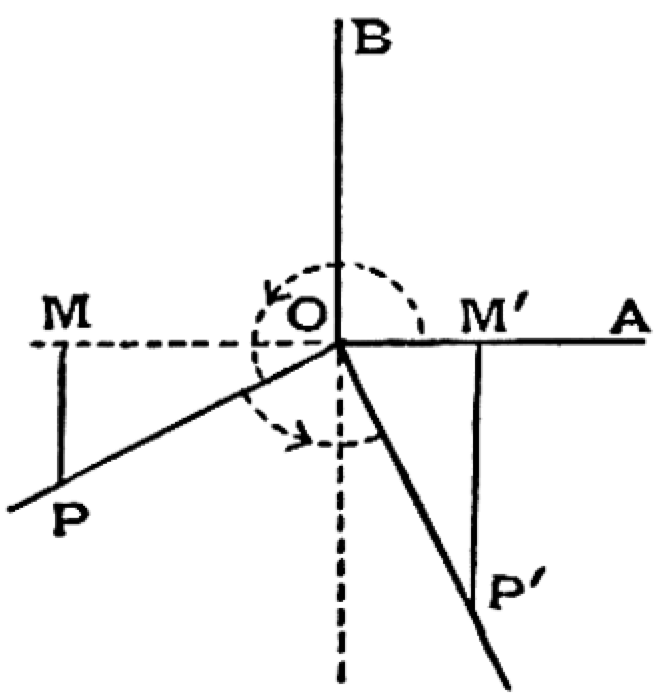
$$OM' = -OM, \text{ and } M'P' = +MP.$$

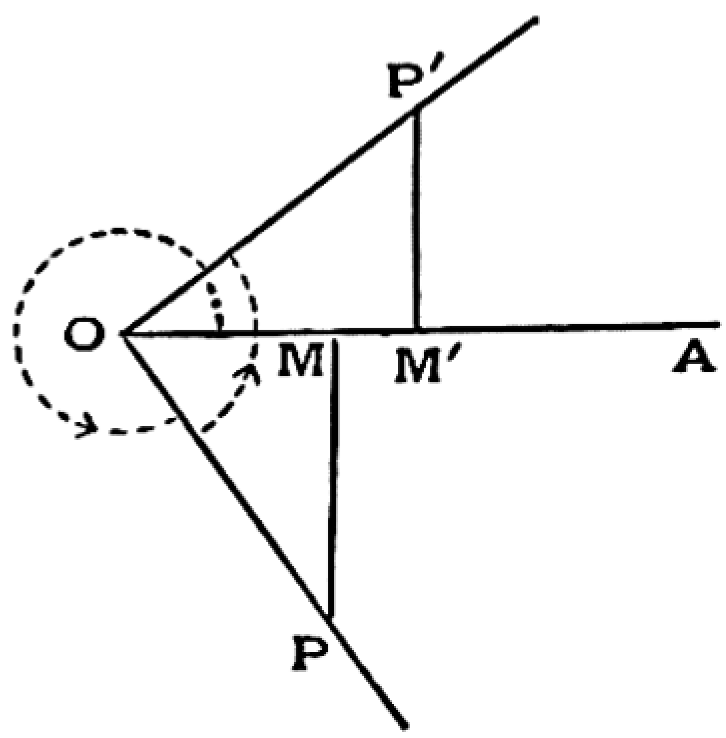
Hence we have

$$\sin(180^\circ - \theta) = \sin AOP' = \frac{M'P'}{OP'} = \frac{MP}{OP} = \sin\theta,$$

$$\cos(180^\circ - \theta) = \cos AOP' = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos\theta.$$

180. *To find the trigonometrical ratios of the angle* $(90^\circ + \theta)$ *in terms of those of* θ, *for all values of* θ.





Let the revolving line, starting from OA, trace out any angle θ and let OP be the position of the revolving line then, so that the angle AOP is θ.

Let the revolving line turn through a right angle from OP in the positive direction to the position OP', so that the angle AOP' is $(90^\circ + \theta)$.

Take OP' equal to OP and draw PM and $P'M'$ perpendicular to AO, produced if necessary. In each figure, since POP' is a right angle, the sum of the angles MOP and $P'OM'$ is always a right angle.

Hence $\angle MOP = 90^\circ - \angle P'OM' = \angle OP'M'$.

The two triangles MOP and $M'P'O$ are therefore equal in all respects.

Hence OM and $M'P'$ are numerically equal, as also MP and OM' are numerically equal.

In each figure, OM and $M'P'$ have the same sign, whilst MP and OM' have the opposite sign, so that

$$M'P' = +OM, \text{ and } OM' = -MP.$$

We therefore have

$$\mathbf{sin}\,(\mathbf{90^\circ}+\boldsymbol{\theta}) = \sin AOP' = \frac{M'P'}{OP'} = \frac{OM}{OP} = \mathbf{cos}\,\boldsymbol{\theta},$$

$$\mathbf{cos}\,(\mathbf{90^\circ}+\boldsymbol{\theta}) = \cos AOP' = \frac{OM'}{OP'} = \frac{-MP}{OP} = -\mathbf{sin}\,\boldsymbol{\theta}.$$

So for the other functions as in Art. 40.

181. *To find the trigonometrical ratios of* $(180^\circ+\theta)$ *in terms of those of* θ, *for all values of* θ.

The required relations may be obtained geometrically, as in the previous articles. The figures for this proposition are easily obtained and are left as an example for the student.

We easily obtain

$$\mathbf{sin}\,(\mathbf{180^\circ}+\boldsymbol{\theta}) = -\mathbf{sin}\,\boldsymbol{\theta},$$

and

$$\mathbf{cos}\,(\mathbf{180^\circ}+\boldsymbol{\theta}) = -\mathbf{cos}\,\boldsymbol{\theta}.$$

Hence

$$\tan(180^\circ+\theta) = \tan\theta,$$

$$\cot(180^\circ+\theta) = \cot\theta,$$

$$\sec(180^\circ+\theta) = -\sec\theta,$$

and

$$\operatorname{cosec}(180^\circ+\theta) = -\operatorname{cosec}\theta.$$

182. *To find the trigonometrical ratios of an angle* $(360^\circ+\theta)$ *in terms of those of* θ, *for all values of* θ.

In whatever position the revolving line may be when it has described any angle θ, it will be in exactly the same position when it has made one more complete revolution in the positive direction, *i.e.* when it has described an angle $360^\circ+\theta$.

Hence the trigonometrical ratios for an angle $360° + \theta$ are the same as those for θ.

It follows that the addition or subtraction of $360°$, or any multiple of $360°$, to or from any angle does not alter its trigonometrical ratios.

183. From the theorems of this chapter it follows that the trigonometrical ratios of any angle whatever can be reduced to the determination of the trigonometrical ratios of an angle which lies between $0°$ and $45°$.

For example,

$$\sin 1765° = \sin[4 \times 360° + 325°] = \sin 325° \qquad \text{(Art. 182)}$$

$$= \sin(180° + 145°) = -\sin 145° \qquad \text{(Art. 181)}$$

$$= -\sin(180° - 35°) = -\sin 35° \qquad \text{(Art. 179);}$$

$$\tan 1190° = \tan(3 \times 360° + 110°) = \tan 110° \qquad \text{(Art. 182)}$$

$$= \tan(90° + 20°) = -\cot 20° \qquad \text{(Art. 180);}$$

and

$$\operatorname{cosec}(-1465°) = -\operatorname{cosec} 1465° \qquad \text{(Art. 177)}$$

$$= -\operatorname{cosec}(4 \times 360° + 25°) = -\operatorname{cosec} 25° \qquad \text{(Art. 182).}$$

Similarly any other such large angles may be treated. First, multiples of $360°$ should be subtracted until the angle lies between $0°$ and $360°$; if it be then greater than $180°$, it should be reduced by $180°$; if then greater than $90°$, the formulae of Arts. 179 or 180 should be used, and finally, if necessary, the formulae of Art. 178 applied.

EXAMPLES. XXXVII.

Prove that

1. $\sin 420° \cos 390° + \cos(-300°) \sin(-330°) = 1$.

2. $\cos 570° \sin 510° - \sin 330° \cos 390° = 0$.

and **3.** $\tan 225° \cot 405° + \tan 765° \cot 675° = 0$.

What are the values of $\cos A - \sin A$ and $\tan A + \cot A$, when A has the values

4. $\frac{7\pi}{4}$ and **5.** $\frac{11\pi}{3}$?

What values between $0°$ and $360°$ may A have when

6. $\sin A = \frac{1}{\sqrt{2}}$, **7.** $\cos A = -\frac{1}{2}$, **8.** $\tan A = -1$,

9. $\cot A = -\sqrt{3}$, **10.** $\sec A = -\frac{2}{\sqrt{3}}$ and **11.** $\operatorname{cosec} A = -2$?

Express in terms of the ratios of a positive angle, which is less than 45°, the quantities

12. $\cos 287°$. **13.** $\tan(-246°)$. **14.** $\sin 843°$.

15. $\cos(-928°)$. **16.** $\tan 1145°$. **17.** $\cos 1410°$.

18. $\cot(-1054°)$. **19.** $\sec 1327°$ and **20.** $\operatorname{cosec}(-756°)$.

What sign has $\sin A + \cos A$ for the following values of A?

21. 278°. **22.** −356° and **23.** −1125°.

What sign has $\sin A - \cos A$ for the following values of A?

24. 215°. **25.** 825°. **26.** −634° and **27.** −457°.

28. Find the sines and cosines of all angles in the first four quadrants whose tangents are equal to $\cos 135°$.

Prove that

29. $\sin(270°+A) = -\cos A$, and $\tan(270°+A) = -\cot A$.

30. $\cos(270°-A) = -\sin A$, and $\cot(270°-A) = \tan A$.

31. $\cos A + \sin(270°+A) - \sin(270°-A) + \cos(180°+A) = 0$.

32. $\sec(270°-A)\sec(90°-A) - \tan(270°-A)\tan(90°+A) + 1 = 0$.

33. $\cot A + \tan(180°+A) + \tan(90°+A) + \tan(360°-A) = 0$.

184. The Addition and Subtraction Theorems can now be shewn to be true for all angles by the aid of the general theorems of this Chapter.

Addition Theorems. In Art. 44 these theorems were proved for the case in which A and B were positive acute angles.

Let now $A_1 = 90° + A$, so that, by Art 180, we have

$$\sin A_1 = \cos A, \text{ and } \cos A_1 = -\sin A.$$

Then

$$\sin(A_1 + B) = \sin\{90° + (A+B)\} = \cos(A+B), \text{ by Art. 180,}$$

$$= \cos A\cos B - \sin A\sin B = \sin A_1\cos B + \cos A_1\sin B.$$

Also $\cos(A_1 + B) = \cos[90° + (A+B)] = -\sin(A+B)$

$$= -\sin A\cos B - \cos A\sin B = \cos A_1\cos B - \sin A_1\sin B.$$

Similarly, we may proceed if B be increased by 90°.

Hence the formulae of Art. 44 are true if either A or B be increased by $90°$, *i.e.* they are true if the component angles lie between $0°$ and $180°$.

Similarly, by putting $A_2 = 90° + A_1$, we can prove the truth of the theorems when either or both of the component angles have values between $0°$ and $270°$.

By proceeding in this way, we see that the theorems are true universally for any positive angles.

Again, if either theorem is true for *any* positive angles A and B it is true for the corresponding negative angles.

For $\qquad \sin(-A-B) = -\sin(A+B) \qquad$ (Art. 177)

$$= -\sin A \cos B - \cos A \sin B$$

$$= \sin(-A)\cos(-B) + \cos(-A)\sin(-B) \quad \text{(Art. 177).}$$

So for $\cos(-A-B)$.

Hence the theorems are true for the negative angles $-A$, $-B$.

185. Subtraction Theorems. In Art. 45 these theorems were proved for the case in which A and B were positive acute angles.

Then, putting $A_1 = 90° + A$, we have

(since $\sin A_1 = \cos A$, and $\cos A_1 = -\sin A$),

$$\sin(A_1 - B) = \sin[90° + (A-B)] = \cos(A-B) \quad \text{(Art. 180)}$$
$$= \cos A \cos B + \sin A \sin B$$
$$= \sin A_1 \cos B - \cos A_1 \sin B.$$

Also

$$\cos(A_1 - B) = \cos[90° + (A-B)] = -\sin(A-B) \quad \text{(Art. 180)}$$
$$= -\sin A \cos B + \cos A \sin B$$
$$= \cos A_1 \cos B + \sin A_1 \sin B.$$

Similarly we may proceed if B be increased by $90°$.

Hence the theorem is true for all angles which are not greater than two right angles.

So, by putting $A_2 = 90° + A_1$, we may shew the theorems to be true for all angles less than three right angles, and so on.

Hence, by proceeding in this manner, we may shew that the theorems are true for all positive angles whatever.

Similarly, as in the last article, for all negative angles.

186. Since the Addition and Subtraction Theorems have now been proved true for all angles, the general formulae of Chapters VI., VII. etc. which are, or can be, deduced from them are necessarily true for all angles.

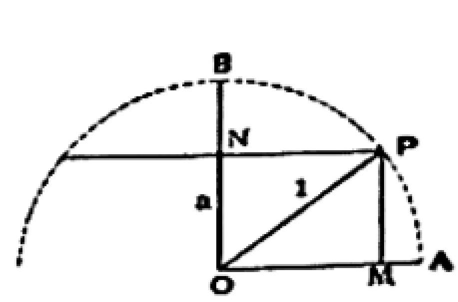
CHAPTER XVI.

GENERAL EXPRESSIONS FOR ALL ANGLES HAVING A GIVEN TRIGONOMETRICAL RATIO. SOLUTION OF TRIGONOMETRICAL EQUATIONS.

187. *To construct the least positive angle whose sine is equal to a, where a is a proper fraction.*

Let OA be the initial line, and let OB be drawn in the positive direction perpendicular to OA.

Measure off along OB a distance ON which is equal to a units of length. [If a be negative the point N will lie in BO produced.]



Through N draw NP parallel to OA. With centre O, and radius equal to the unit of length, describe a circle and let it meet NP in P.

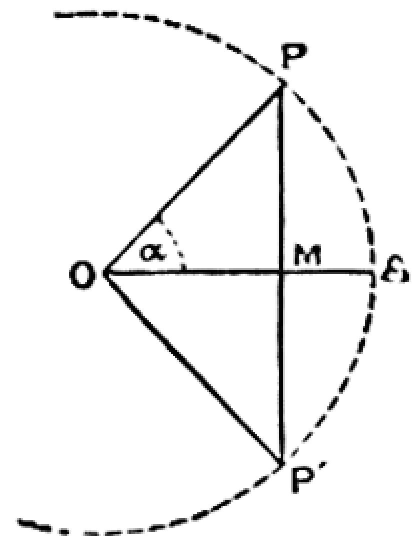
Draw PM perpendicular to OA.

Since $$\sin AOP = \frac{MP}{OP} = \frac{ON}{OP} = \frac{a}{1} = a,$$

therefore AOP is the angle required.

188. *To construct the least positive angle whose cosine is equal to b, where b is a proper fraction.*

Along the initial line measure off a distance OM equal to b and draw MP perpendicular to OA. [If b be negative, M will lie on the other side of O in the line AO produced.]



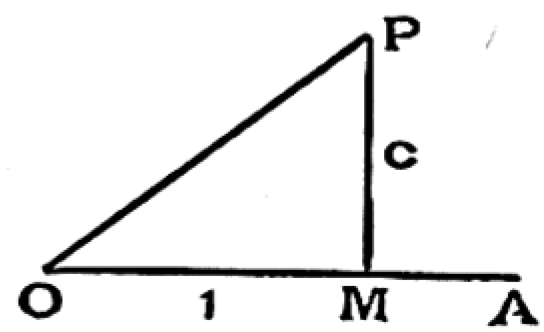
With centre O, and radius equal to unity, describe a circle and let it meet MP in P.

Then AOP is the angle required. For

$$\cos AOP = \frac{OM}{OP} = \frac{b}{1} = b.$$

189. *To construct the least positive angle whose tangent is equal to c.*

Along the initial line measure off OM equal to unity, and erect a perpendicular MP. Measure off MP equal to c.



Then

$$\tan AOP = \frac{MP}{OM} = c,$$

so that AOP is the required angle.

190. It is clear from the definition given in Art. 35, that, when an angle is given, so also is its sine. The converse statement is not correct; there is more than one angle having a given sine; for example, the angles 30°, 150°, 390°, −210°, ... all have their sine equal to $\frac{1}{2}$.

Hence, when the sine of an angle is given, we do not definitely know the angle; all we know is that the angle is one out of a large number of angles.

Similar statements are true if the cosine, tangent, or any other trigonometrical function of the angle be given.

Hence, simply to give *one* of the trigonometrical functions of an angle does not determine it without ambiguity.

191. Suppose we know that the revolving line OP coincides with the initial line OA. All we know is that the revolving line has made 0, or 1, or 2, or 3,... complete revolutions, either positive or negative.

But when the revolving line has made one complete revolution, the angle it has described is equal to 2π radians.

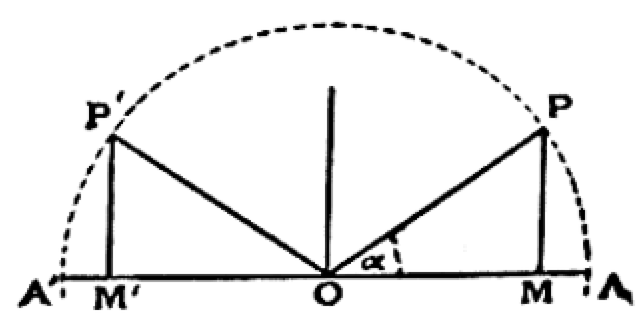
Hence, when the revolving line OP coincides with the initial line OA, the angle that it has described is 0, or 1, or 2, or 3, ... times 2π radians, in either the positive or negative directions, *i.e.* either 0, or $\pm 2\pi$, or $\pm 4\pi$, or $\pm 6\pi$, ... radians.

This is expressed by saying that when the revolving line coincides with the initial line the angle it has described is $\mathbf{2n\pi}$, where n is either zero or some positive or negative integer.

192. Theorem. *To find a general expression to include all angles which have the same sine.*

Let AOP be any angle having the given sine, and let it be denoted by α.

Draw PM perpendicular to OA and produce MO to M', making OM' equal to MO, and draw $M'P'$ parallel and equal to MP.



The triangles $M'OP'$, MOP are then equal in all respects so that $\angle A'OP' = \angle AOP = \alpha$, and $\therefore \angle AOP' = \pi - \alpha$.

When the revolving line is in either of the positions OP or OP', and in no other position, the sine of the angle traced out is equal to the given sine.

When the revolving line is in the position OP, it has either described the angle AOP, or has made a whole number of complete revolutions and then described an angle α, *i.e.*, by the last article, it has described an angle equal to

$$2r\pi + \alpha \qquad \text{.........................(1)},$$

where r is zero or some positive or negative integer.

When the revolving line is in the position OP', it has, similarly, described an angle $2r\pi + AOP'$, *i.e.* an angle $2r\pi + \pi - \alpha$,

i.e. $$(2r+1)\pi - \alpha \text{.................... (2)},$$

where r is zero or some positive or negative integer.

All these angles will be found to be included in the expression

$$\mathbf{n\pi + (-1)^n \alpha} \text{.................... (3)},$$

where n is zero or a positive or negative integer.

For, when $n = 2r$, since $(-1)^{2r} = +1$, the expression (3) gives $2r\pi + \alpha$, which is the same as the expression (1).

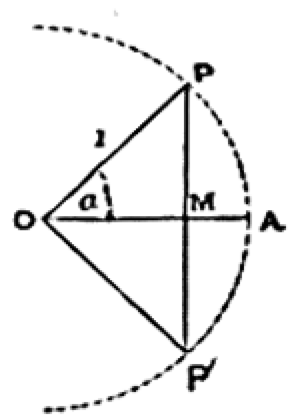
Also, when $n = 2r + 1$, since $(-1)^{2r+1} = -1$, the expression (3) gives $(2r+1)\pi - \alpha$, which is the same as the expression (2).

Cor. Since all angles which have the same sine have also the same cosecant, the expression (3) includes all angles which have the same cosecant as α.

193. Theorem. *To find a general expression to include all angles which have the same cosine.*

Let AOP be any angle having the given cosine, and let it be denoted by α.

Draw PM perpendicular to OA and produce it to P', making MP' equal to PM.



When the revolving line is in the position OP or OP', and in no other position, then, as in Art. 188, the cosine of the angle traced out is equal to the given cosine.

When the revolving line is in the position OP, it has made a whole number of complete revolutions and then described an angle α, *i.e.* it has described an angle $2n\pi + \alpha$, where n is zero or some positive or negative integer.

When the revolving line is in the position OP', it has made a whole number of complete revolutions and then described an angle $-\alpha$, *i.e.* it has described an angle $2n\pi - \alpha$.

All these angles are included in the expression

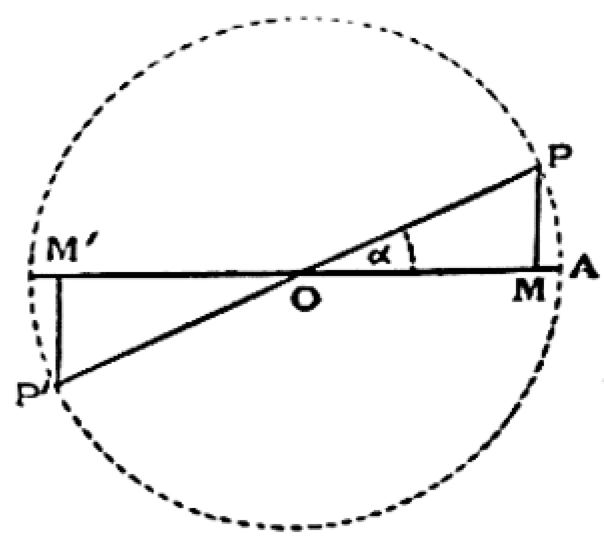
$$\mathbf{2n\pi \pm \alpha} \dots\dots\dots\dots\dots\dots\dots (1),$$

where n is zero or some positive or negative integer.

Cor. The expression (1) includes all angles having the same secant as α.

194. Theorem. *To find a general expression for all angles which have the same tangent.*

Let AOP be any angle having the given tangent, and let it be denoted by α.



Produce PO to P', making OP' equal to OP, and draw $P'M'$ perpendicular to OM'.

As in Art. 181, the angles AOP and AOP' have the same tangent; also the angle $AOP' = \pi + \alpha$.

When the revolving line is in the position OP, it has described a whole number of complete revolutions and then turned through an angle α, *i.e.* it has described an angle

$$2r\pi + \alpha \dots\dots\dots\dots\dots\dots\dots (1),$$

where r is zero or some positive or negative integer.

When the revolving line is in the position OP', it has similarly described an angle $2r\pi + (\pi + \alpha)$,

i.e. $$(2r + 1)\pi + \alpha \dots\dots\dots\dots\dots\dots\dots (2).$$

All these angles are included in the expression

$$\mathbf{n\pi + \alpha} \dots\dots\dots\dots\dots\dots\dots (3),$$

where n is zero or some positive or negative integer.

For, when n is even, $(= 2r$ say), the expression (3) gives the same angles as the expression (1).

Also when n is odd, $(= 2r + 1$ say), it gives the same angles as the expression (2).

Cor. The expression (3) includes all angles which have the same cotangent as α.

195. In Arts. 192, 193 and 194 the angle α is any angle satisfying the given condition. In practical examples it is, in general, desirable to take α as the smallest positive angle which is suitable.

Ex. 1. *Write down the general expression for all angles,*

(1) *whose sine is equal to* $\frac{\sqrt{3}}{2}$,

(2) *whose cosine is equal to* $-\frac{1}{2}$,

and (3) *whose tangent is equal to* $\frac{1}{\sqrt{3}}$.

(1) The smallest angle, whose sine is $\frac{\sqrt{3}}{2}$, is $60°$, *i.e.* $\frac{\pi}{3}$.

Hence, by Art. 192, the general expression for all the angles which have this sine is

$$n\pi + (-1)^n \frac{\pi}{3}.$$

(2) The smallest positive angle, whose cosine is $-\frac{1}{2}$,

is $120°$, *i.e.* $\frac{2\pi}{3}$.

Hence, by Art. 193, the general expression for all the angles which have this cosine is

$$2n\pi \pm \frac{2\pi}{3}.$$

(3) The smallest positive angle, whose tangent is $\frac{1}{\sqrt{3}}$,

is $30°$, *i.e.* $\frac{\pi}{6}$.

Hence, by Art. 194, the general expression for all the angles which have this tangent is

$$n\pi + \frac{\pi}{6}.$$

Ex. 2. *What is the most general value of θ satisfying the equation* $\sin^2\theta = \frac{1}{4}$?

We have $\quad \cos 2\theta = 1 - 2\sin^2\theta = 1 - \frac{1}{2} = \frac{1}{2} = \cos\frac{\pi}{3}$

$$\therefore\ 2\theta = 2n\pi \pm \frac{\pi}{3} \qquad \text{[Art. 193]},$$

i.e.

$$\theta = n\pi \pm \frac{\pi}{6}.$$

Ex. 3. *What is the most general value of θ which satisfies both of the equations* $\sin\theta = -\frac{1}{2}$ *and* $\tan\theta = \frac{1}{\sqrt{3}}$?

Considering only angles between 0° and 360°, the only values of θ, when $\sin\theta = -\frac{1}{2}$, are 210° and 330°. Similarly, the only values of θ, when $\tan\theta = \frac{1}{\sqrt{3}}$, are 30° and 210°.

The only value of θ, between 0° and 360°, satisfying both conditions is therefore 210°, *i.e.* $\frac{7\pi}{6}$.

The most general value is hence obtained by adding any multiple of four right angles to this angle, and hence is $2n\pi + \frac{7\pi}{6}$, where n is any positive or negative integer.

EXAMPLES. XXXVIII.

What are the most general values of θ which satisfy the equations:

1. $\sin\theta = \frac{1}{2}$. 2. $\sin\theta = -\frac{\sqrt{3}}{2}$. 3. $\sin\theta = \frac{1}{\sqrt{2}}$.

4. $\cos\theta = -\frac{1}{2}$. 5. $\cos\theta = \frac{\sqrt{3}}{2}$. 6. $\cos\theta = -\frac{1}{\sqrt{2}}$.

7. $\tan\theta = \sqrt{3}$. 8. $\tan\theta = -1$. 9. $\cot\theta = 1$.

10. $\sec\theta = 2$. 11. $\operatorname{cosec}\theta = \frac{2}{\sqrt{3}}$. 12. $\sin^2\theta = 1$.

13. $\cos^2\theta = \frac{1}{4}$. 14. $\tan^2\theta = \frac{1}{3}$. 15. $\sin^2\theta = \sin^2\alpha$.

16. $2\cot^2\theta = \operatorname{cosec}^2\theta$. 17. $\sec^2\theta = \frac{4}{3}$?

18. What is the most general value of θ that satisfies both of the equations

$$\cos\theta = -\frac{1}{\sqrt{2}} \text{ and } \tan\theta = 1?$$

19. What is the most general value of θ that satisfies both of the equations

$$\cot\theta = -\sqrt{3} \text{ and } \operatorname{cosec}\theta = -2?$$

20. Find the angles between 0° and 360° which have respectively (1) their sines equal to $\frac{\sqrt{3}}{2}$, (2) their cosines equal to $-\frac{1}{2}$, and (3) their tangents equal to $\frac{1}{\sqrt{3}}$.

21. Given the angle x construct the angle y if (1) $\sin y = 2\sin x$, (2) $\tan y = 3\tan x$, (3) $\cos y = \frac{1}{2}\cos x$, and (4) $\sec y = \operatorname{cosec} x$.

196. An equation involving the trigonometrical ratios of an unknown angle is called a trigonometrical equation.

The equation is not completely solved unless we obtain an expression for all the angles which satisfy it.

Some elementary types of equations are solved in the following article.

197. Ex. 1. *Solve the equation* $2\sin^2 x + \sqrt{3}\cos x + 1 = 0$.

The equation may be written

$$2 - 2\cos^2 x + \sqrt{3}\cos x + 1 = 0,$$

i.e.
$$2\cos^2 x - \sqrt{3}\cos x - 3 = 0,$$

i.e.
$$(\cos x - \sqrt{3})(2\cos x + \sqrt{3}) = 0.$$

The equation is therefore satisfied by $\cos x = \sqrt{3}$, or $\cos x = -\frac{\sqrt{3}}{2}$.

Since the cosine of an angle cannot be numerically greater than unity, the first factor gives no solution.

The smallest positive angle, whose cosine is $-\frac{\sqrt{3}}{2}$, is $150°$, *i.e.* $\frac{5\pi}{6}$.

Hence the most general value of the angle, whose cosine is $-\frac{\sqrt{3}}{2}$, is $2n\pi \pm \frac{5\pi}{6}$. (Art. 193.)

This is the general solution of the given equation.

Ex. 2. *Solve the equation* $\tan 5\theta = \cot 2\theta$.

The equation may be written

$$\tan 5\theta = \tan\left(\frac{\pi}{2} - 2\theta\right).$$

Now the most general value of the angle, that has the same tangent as $\frac{\pi}{2} - 2\theta$, is, by Art. 194, $n\pi + \frac{\pi}{2} - 2\theta$, where n is any positive or negative integer.

The most general solution of the equation is therefore

$$5\theta = n\pi + \frac{\pi}{2} - 2\theta.$$

$$\therefore\ \theta = \frac{1}{7}\left(n\pi + \frac{\pi}{2}\right),$$

where n is any integer.

EXAMPLES. XXXIX.

Solve the equations:

1. $\cos^2\theta - \sin\theta - \frac{1}{4} = 0.$

2. $2\sin^2\theta + 3\cos\theta = 0.$

3. $2\sqrt{3}\cos^2\theta = \sin\theta.$

4. $\cos\theta + \cos^2\theta = 1.$

5. $4\cos\theta - 3\sec\theta = 2\tan\theta.$

6. $\cos 2\theta + 4\cos\theta = \frac{3}{2}.$

7. $\tan^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 0.$

8. $\cot^2\theta + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)\cot\theta + 1 = 0.$

9. $\cot\theta - ab\tan\theta = a - b.$

10. $\tan^2\theta + \cot^2\theta = 2.$

11. $\sec\theta - 1 = (\sqrt{2} - 1)\tan\theta.$

12. $3(\sec^2\theta + \tan^2\theta) = 5.$

13. $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta.$

14. $4\cos^2\theta + \sqrt{3} = 2(\sqrt{3}+1)\cos\theta.$

15. $3\cos 2\theta + 4\sin\theta = 1.$

16. $\sin 5\theta = \frac{1}{\sqrt{2}}.$

17. $\sin 9\theta = \sin\theta.$

18. $\sin 3\theta = \sin 2\theta.$

19. $\cos m\theta = \cos n\theta.$

20. $\sin 2\theta = \cos 3\theta.$

21. $\cos 5\theta = \cos 4\theta.$

22. $\cos m\theta = \sin n\theta.$

23. $\cot\theta = \tan 8\theta.$

24. $\cot\theta = \tan n\theta.$

25. $\tan 2\theta = \tan\frac{2}{\theta}.$

26. $\tan 2\theta\tan\theta = 1.$

27. $\tan^2 3\theta = \cot^2\alpha.$

28. $\tan 3\theta = \cot\theta.$

29. $\tan^2 3\theta = \tan^2\alpha.$

30. $3\tan^2\theta = 1.$

31. $\tan mx + \cot nx = 0.$

32. $\sin(\theta - \phi) = \frac{1}{2}$, and $\cos(\theta + \phi) = \frac{1}{2}.$

33. $\cos(2x + 3y) = \frac{1}{2}$, and $\cos(3x + 2y) = \frac{\sqrt{3}}{2}.$

34. If $\tan^2\theta = \frac{5}{4}$, find versin θ and explain the double result.

198. The Addition and Subtraction Theorems may be used to solve some kinds of trigonometrical equations.

Ex. *Solve the equation*

$$\sin x + \sin 5x = \sin 3x.$$

By the formulae of Art. 48, the equation is

$$2 \sin 3x \cos 2x = \sin 3x.$$

$$\therefore \ \sin 3x = 0, \text{ or } 2 \cos 2x = 1.$$

If $\sin 3x = 0$, then $3x = n\pi$.

If $\cos 2x = \frac{1}{2}$, then $2x = 2n\pi \pm \frac{\pi}{3}$.

Hence $x = \frac{n\pi}{3}$, or $n\pi \pm \frac{\pi}{6}$.

199. *To solve an equation of the form*

$$a \cos \theta + b \sin \theta = c.$$

Divide both sides of the equation by $\sqrt{a^2 + b^2}$, so that it may be written

$$\frac{a}{\sqrt{a^2 + b^2}} \cos \theta + \frac{b}{\sqrt{a^2 + b^2}} \sin \theta = \frac{c}{\sqrt{a^2 + b^2}}.$$

Find from the table of tangents the angle whose tangent is $\frac{b}{a}$ and call it α.

Then $\tan \alpha = \frac{b}{a}$, so that

$$\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}, \text{ and } \cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}.$$

The equation can then be written

$$\cos \alpha \cos \theta + \sin \alpha \sin \theta = \frac{c}{\sqrt{a^2 + b^2}},$$

i.e.

$$\cos (\theta - \alpha) = \frac{c}{\sqrt{a^2 + b^2}}.$$

Next find from the tables, or otherwise, the angle β whose cosine is $\frac{c}{\sqrt{a^2 + b^2}}$,

so that

$$\cos \beta = \frac{c}{\sqrt{a^2 + b^2}}.$$

[N.B. This can only be done when c is $< \sqrt{a^2 + b^2}$.]

The equation is then $\cos(\theta - \alpha) = \cos\beta$.

The solution of this is $\theta - \alpha = 2n\pi \pm \beta$, so that

$$\theta = 2n\pi + \alpha \pm \beta,$$

where n is any integer.

Angles, such as α and β, which are introduced into trigonometrical work to facilitate computation are called **Subsidiary Angles.**

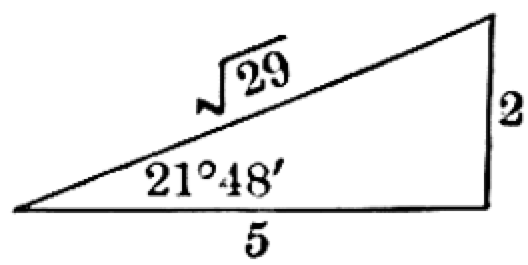
As a numerical example let us solve the equation.

$$5\cos\theta - 2\sin\theta = 2,$$

given that $\tan 21°\,48' = \frac{2}{5}$.

Dividing both sides of the equation by

$$\sqrt{5^2+2^2}, \text{ i.e. } \sqrt{29},$$



we have

$$\frac{5}{\sqrt{29}}\cos\theta - \frac{2}{\sqrt{29}}\sin\theta = \frac{2}{\sqrt{29}}.$$

Hence $\cos\theta\cos 21°\,48' - \sin\theta\sin 21°\,48'$

$$= \sin 21°\,48' = \sin(90° - 68°\,12')$$

$$= \cos 68°\,12'.$$

$$\therefore \cos(\theta + 21°\,48') = \cos 68°\,12'.$$

Hence $\theta + 21°\,48' = 2n \times 180° \pm 68°\,12'$. (Art. 193.)

$$\therefore \theta = 2n \times 180° - 21°\,48' \pm 68°\,12'$$

$$= 2n \times 180° - 90°, \text{ or } 2n \times 180° + 46°\,24',$$

where n is any integer.

200. **Aliter.** The equation of the last article may be solved in another way.

For let $t \equiv \tan\frac{\theta}{2}$,

so that $$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2t}{1+t^2},$$

and $$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-t^2}{1+t^2}. \quad \text{(Art. 61.)}$$

The equation then becomes

$$a\frac{1-t^2}{1+t^2}+b\frac{2t}{1+t^2}=c,$$

so that $$t^2(c+a)-2bt+c-a=0.$$

This is a quadratic equation giving two values for t and hence two values for $\tan\frac{\theta}{2}$.

Thus, the example of this article gives

$$7t^2+4t-3=0,$$

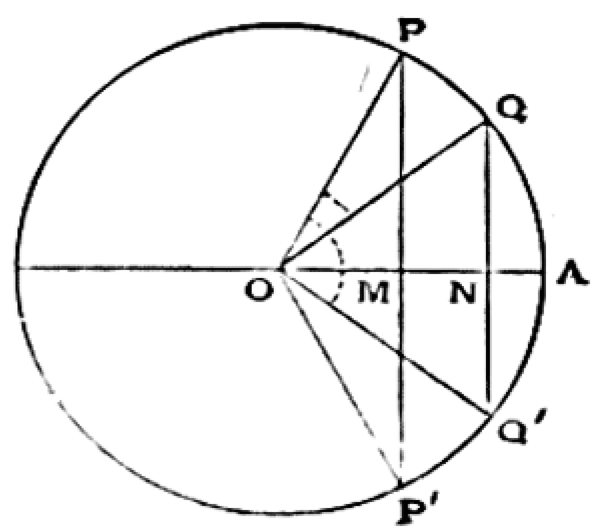
so that $$t=-1 \text{ or } \frac{3}{7}=-1 \text{ or } \cdot 4286$$

$$=\tan(-45^\circ) \text{ or } \tan 23^\circ 12' \text{ (from the tables).}$$

Hence $$\frac{\theta}{2}=n\,.\,180^\circ-45^\circ, \text{ or } n\,.\,180^\circ+23^\circ 12',$$

i.e. $$\theta=n\,.\,360^\circ-90^\circ, \text{ or } n\,.\,360^\circ+46^\circ 24'.$$

201. The solution of Art. 199 may be illustrated graphically as follows:



Measure OM along the initial line equal to a, and MP perpendicular to it, and equal to b. The angle MOP is then the angle whose tangent is $\frac{b}{a}$, *i.e.* α.

With centre O and radius OP, *i.e.* $\sqrt{a^2+b^2}$, describe a circle, and measure ON along the initial line equal to c.

Draw QNQ' perpendicular to ON to meet the circle in Q and Q'; the angles NOQ and $Q'ON$ are therefore each equal to β.

The angle QOP is therefore $\alpha-\beta$ and $Q'OP$ is $\alpha+\beta$.

Hence the solutions of the equation are respectively

$$2n\pi+QOP \text{ and } 2n\pi+Q'OP.$$

The construction clearly fails if c be $>\sqrt{a^2+b^2}$, for then the point N would fall outside the circle.

EXAMPLES. XL.

Solve the equations

1. $\sin\theta+\sin 7\theta=\sin 4\theta.$

2. $\cos\theta+\cos 7\theta=\cos 4\theta.$

3. $\cos\theta+\cos 3\theta=2\cos 2\theta.$

4. $\sin 4\theta-\sin 2\theta=\cos 3\theta.$

5. $\cos\theta - \sin 3\theta = \cos 2\theta$.

6. $\sin 7\theta = \sin\theta + \sin 3\theta$.

7. $\cos\theta + \cos 2\theta + \cos 3\theta = 0$.

8. $\sin\theta + \sin 3\theta + \sin 5\theta = 0$.

9. $\sin 2\theta - \cos 2\theta - \sin\theta + \cos\theta = 0$.

10. $\sin(3\theta + a) + \sin(3\theta - a) + \sin(a - \theta) - \sin(a + \theta) = \cos a$.

11. $\cos(3\theta + a)\cos(3\theta - a) + \cos(5\theta + a)\cos(5\theta - a) = \cos 2a$.

12. $\sin 5\theta\cos 3\theta = \sin 9\theta\cos 7\theta$.

13. $\cos 4\theta\cos\theta = \sin 6\theta\sin 3\theta$.

14. $\sin m\theta + \sin n\theta = 0$.

15. $\cos m\theta + \cos n\theta = 0$.

16. $\sin^2 n\theta - \sin^2(n-1)\theta = \sin^2\theta$.

17. $\sin 3\theta + \cos 2\theta = 0$.

18. $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$.

19. $\sin\theta + \cos\theta = \sqrt{2}$.

20. $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$.

21. $\sin x + \cos x = \sqrt{2}\cos A$.

22. $5\sin\theta + 2\cos\theta = 5$ (given $\tan 21°48' = \cdot 4$).

23. $6\cos x + 8\sin x = 9$ (given $\tan 53°8' = 1\frac{1}{3}$ and $\cos 25°50' = \cdot 9$).

24. $1 + \sin^2\theta = 3\sin\theta\cos\theta$ (given $\tan 71°34' = 3$).

25. $\operatorname{cosec}\theta = \cot\theta + \sqrt{3}$.

26. $\operatorname{cosec} x = 1 + \cot x$.

27. $(2 + \sqrt{3})\cos\theta = 1 - \sin\theta$.

28. $\tan\theta + \sec\theta = \sqrt{3}$.

29. $\cos 2\theta = \cos^2\theta$.

30. $4\cos\theta - 3\sec\theta = \tan\theta$.

31. $\cos 2\theta + 3\cos\theta = 0$.

32. $\cos 3\theta + 2\cos\theta = 0$.

33. $\cot\theta - \tan\theta = 2$.

34. $4\cot 2\theta = \cot^2\theta - \tan^2\theta$.

35. $3\tan(\theta - 15°) = \tan(\theta + 15°)$.

36. $\tan\theta + \tan 2\theta + \tan 3\theta = 0$.

37. $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$.

38. $\sin 3a = 4\sin a\sin(x + a)\sin(x - a)$.

CHAPTER XVII.

INVERSE CIRCULAR FUNCTIONS.

202. IF $\sin\theta = a$, where a is a known quantity, we know, from Art. 190, that θ is not definitely known. We only know that θ is some one of a definite series of angles.

The symbol "$\sin^{-1}a$" is used to denote the *smallest* angle, whether positive or negative, that has a for its sine.

The symbol "$\sin^{-1}a$" is read in words as "sine minus one a," and must be carefully distinguished from $\dfrac{1}{\sin a}$ which would be written, if so desired, in the form $(\sin a)^{-1}$.

It will therefore be carefully noted that "$\sin^{-1}a$" is an **angle,** and denotes the **smallest numerical** angle whose sine is a.

When a is positive, $\sin^{-1}a$ clearly lies between 0° and 90°; when a is negative, it lies between $-90°$ and 0°.

Ex. $\sin^{-1}\frac{1}{2} = 30°$; $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -60°$.

So "$\cos^{-1}a$" means the smallest numerical angle whose cosine is a.

When a is positive, there are two angles, one lying between 0° and 90° and the other lying between $-90°$ and 0°, each of which has its cosine equal to a. [For example,

both 30° and $-30°$ have their cosine equal to $\frac{\sqrt{3}}{2}$.] In this case we take the smallest *positive* angle. Hence $\cos^{-1}a$, when a is positive, lies between 0° and 90°.

So $\cos^{-1}a$, when a is negative, lies between 90° and 180°.

Ex. $\cos^{-1}\frac{1}{\sqrt{2}}=45°$; $\cos^{-1}\left(-\frac{1}{2}\right)=120°$.

Similarly, $\tan^{-1}a$ means the smallest angle whose tangent is a.

When a is positive, the angle $\tan^{-1}a$ lies between 0° and 90°; when a is negative, it lies between $-90°$ and 0°.

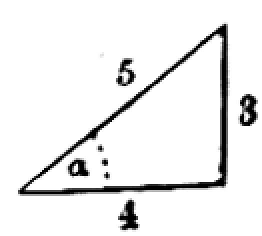
Ex. $\tan^{-1}\sqrt{3}=60°$; $\tan^{-1}(-1)=-45°$.

Similarly, $\cot^{-1}a$, $\operatorname{cosec}^{-1}a$, ... are defined.

These quantities are called Inverse Circular, or Trigonometrical, Functions.

203. **Ex. 1.** *Prove that* $\sin^{-1}\frac{3}{5}-\cos^{-1}\frac{12}{13}=\sin^{-1}\frac{16}{65}$.

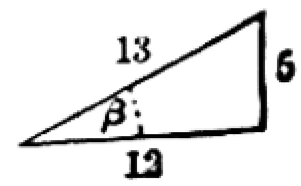
Let $\sin^{-1}\frac{3}{5}=\alpha$, so that $\sin\alpha=\frac{3}{5}$,



and therefore $\cos\alpha=\sqrt{1-\frac{9}{25}}=\frac{4}{5}$.

Let $\cos^{-1}\frac{12}{13}=\beta$, so that $\cos\beta=\frac{12}{13}$,

and therefore $\sin\beta=\sqrt{1-\frac{144}{169}}=\frac{5}{13}$.



Let $\sin^{-1}\frac{16}{65}=\gamma$, so that $\sin\gamma=\frac{16}{65}$.

We have then to prove that

$$\alpha-\beta=\gamma,$$

i.e. to shew that $\sin(\alpha-\beta)=\sin\gamma$.

Now
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$=\frac{3}{5}\cdot\frac{12}{13}-\frac{4}{5}\cdot\frac{5}{13}=\frac{36-20}{65}=\frac{16}{65}=\sin\gamma.$$

Hence the relation is proved.

Ex. 2. *Prove that* $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$.

Let $\tan^{-1}\frac{1}{3}=\alpha$, so that $\tan\alpha=\frac{1}{3}$,

and let $\tan^{-1}\frac{1}{7}=\beta$, so that $\tan\beta=\frac{1}{7}$.

We have then to shew that

$$2\alpha+\beta=\frac{\pi}{4}.$$

Now
$$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}$$
$$=\frac{\frac{2}{3}}{1-\frac{1}{9}}=\frac{6}{8}=\frac{3}{4}.$$

Also,
$$\tan(2\alpha+\beta)=\frac{\tan 2\alpha+\tan\beta}{1-\tan 2\alpha\tan\beta}$$
$$=\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}=\frac{21+4}{28-3}=\frac{25}{25}=1=\tan\frac{\pi}{4}.$$

$$\therefore\ 2\alpha+\beta=\frac{\pi}{4}.$$

Ex. 3. *Prove that*

$$4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\frac{\pi}{4}.$$

Let $\tan^{-1}\frac{1}{5}=\alpha$, so that $\tan\alpha=\frac{1}{5}$.

Then
$$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}=\frac{\frac{2}{5}}{1-\frac{1}{25}}=\frac{5}{12},$$

and
$$\tan 4\alpha=\frac{\frac{10}{12}}{1-\frac{25}{144}}=\frac{120}{119},$$

so that $\tan 4\alpha$ is nearly unity, and 4α therefore nearly $\frac{\pi}{4}$.

Let $4\alpha=\frac{\pi}{4}+\tan^{-1}x$.

$$\therefore \ \frac{120}{119} = \tan\left(\frac{\pi}{4} + \tan^{-1}x\right) = \frac{1+x}{1-x} \quad \textbf{(Art. 54).}$$

$$\therefore \ x = \frac{1}{239}.$$

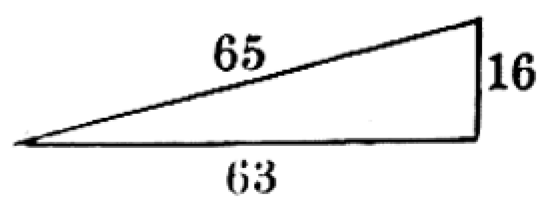
Hence $$4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}.$$

Ex. 4. *Prove that*

$$\cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = \sin^{-1}\frac{3}{5}.$$

Since $65^2 - 63^3 = 16^2$, we have

$$\cos^{-1}\frac{63}{65} = \tan^{-1}\frac{16}{63}.$$



Also, as in Ex. 1, $\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4}$.

We have therefore to shew that

$$\tan^{-1}\frac{16}{63} + 2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{3}{4}.$$

Now

$$\tan\left[2\tan^{-1}\frac{1}{5}\right] = \frac{2\tan\left[\tan^{-1}\frac{1}{5}\right]}{1-\tan^2\left[\tan^{-1}\frac{1}{5}\right]} = \frac{\frac{2}{5}}{1-\frac{1}{25}} = \frac{5}{12},$$

so that $$2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{5}{12}.$$

Thus

$$\tan\left[\tan^{-1}\frac{16}{63} + 2\tan^{-1}\frac{1}{5}\right] = \tan\left[\tan^{-1}\frac{16}{63} + \tan^{-1}\frac{5}{12}\right]$$

$$= \frac{\frac{16}{63} + \frac{5}{12}}{1 - \frac{16}{63}\cdot\frac{5}{12}} = \frac{192+315}{756-80} = \frac{507}{676} = \frac{3}{4},$$

i.e. $$\tan^{-1}\frac{16}{63} + 2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{3}{4}.$$

Ex. 5. *Solve the equation*

$$\tan^{-1}\frac{x+1}{x-1} - \tan^{-1}\frac{x-1}{x} = \tan^{-1}\frac{7}{4}.$$

Taking the tangents of both sides of the equation, we have

$$\frac{\tan\left[\tan^{-1}\frac{x+1}{x-1}\right]-\tan\left[\tan^{-1}\frac{x-1}{x}\right]}{1+\tan\left[\tan^{-1}\frac{x+1}{x-1}\right]\tan\left[\tan^{-1}\frac{x-1}{x}\right]}$$

$$=\tan\left\{\tan^{-1}\frac{7}{4}\right\}=\frac{7}{4},$$

i.e. $$\frac{\frac{x+1}{x-1}-\frac{x-1}{x}}{1+\frac{x+1}{x-1}\frac{x-1}{x}}=\frac{7}{4},$$

i.e. $$\frac{3x-1}{2x^2-x-1}=\frac{7}{4},$$

so that $$x=\frac{3}{2}\text{ or }-\frac{1}{7}.$$

EXAMPLES. XLI.

[*The student should verify the results of some of the following examples (e.g. Nos.* 1—4, 8, 9, 12, 13) *by an accurate graph.*]

Prove that

1. $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}$.

2. $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}=\cos^{-1}\left(\frac{253}{325}\right)$.

3. $\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{3}{5}=\tan^{-1}\frac{27}{11}$.

4. $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$.

5. $\cos^{-1}x=2\sin^{-1}\sqrt{\frac{1-x}{2}}=2\cos^{-1}\sqrt{\frac{1+x}{2}}$.

6. $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}=\sin^{-1}\frac{63}{65}$.

7. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3=45^\circ$.

8. $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$.

9. $\tan^{-1}\frac{2}{3}=\frac{1}{2}\tan^{-1}\frac{12}{5}$.

10. $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\frac{3}{5}$.

11. $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

12. $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}=\frac{\pi}{4}$.

13. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

Solve the equations

14. $\tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}=\theta$

15. $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$.

16. $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$.

17. $\tan^{-1}(x+1)+\cot^{-1}(x-1)=\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{3}{5}$.

18. $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$.

19. $\tan\cos^{-1}x=\sin\cot^{-1}\frac{1}{2}$.

20. $\cot^{-1}x-\cot^{-1}(x+2)=15^\circ$.

21. $\cos^{-1}\frac{x^2-1}{x^2+1}+\tan^{-1}\frac{2x}{x^2-1}=\frac{2\pi}{3}$.

22. $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}$.

23. $\sin^{-1}\frac{5}{x}+\sin^{-1}\frac{12}{x}=\frac{\pi}{2}$.

Draw the graphs of

24. $\sin^{-1}x$. [N.B. If $y=\sin^{-1}x$, then $x=\sin y$ and the graph bears the same relation to OY that the curve in Art. 169 bears to OX.]

25. $\cos^{-1}x$.

26. $\tan^{-1}x$.

27. $\cot^{-1}x$.

28. $\operatorname{cosec}^{-1}x$.

29. $\sec^{-1}x$.

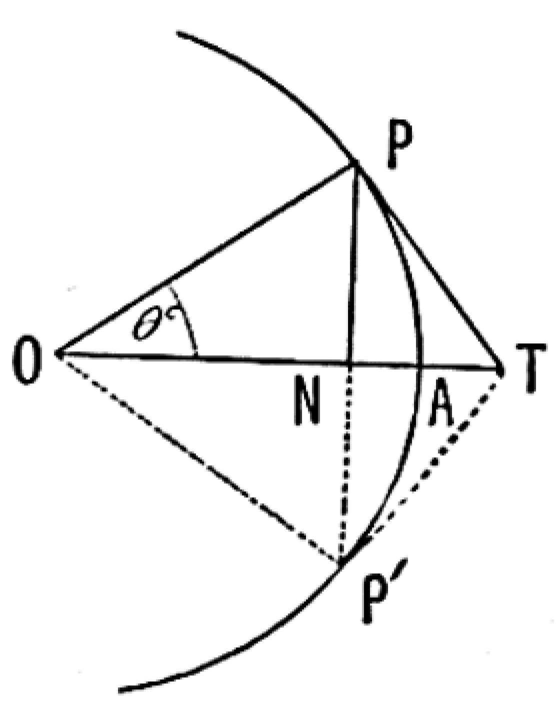
CHAPTER XVIII.

TRIGONOMETRICAL RATIOS OF SMALL ANGLES. AREA OF A CIRCLE. DIP OF THE HORIZON.

204. *If θ be the number of radians in any angle, which is less than a right angle, then $\sin\theta$, θ, and $\tan\theta$ are in ascending order of magnitude.*

Let TOP be any angle which is less than a right angle.

With centre O and any radius OP describe an arc PAP' meeting OT in A.



Draw PN perpendicular to OA, and produce it to meet the arc of the circle in P'.

Draw the tangent PT at P to meet OA in T, and join TP'.

The triangles PON and $P'ON$ are equal in all respects, so that $PN = NP'$ and

$$\text{arc } PA = \text{arc } AP'.$$

Also the triangles TOP and TOP' are equal in all respects, so that

$$TP = TP'.$$

The straight line PP' is less than the arc PAP', so that NP is $<$ arc PA.

We shall *assume* that the arc PAP' is less than the sum of PT and TP', so that arc $PA < PT$.

Hence NP, the arc AP, and PT are in ascending order of magnitude.

Therefore $\frac{NP}{OP}$, $\frac{\text{arc } AP}{OP}$, and $\frac{PT}{OP}$ are in ascending order of magnitude.

But
$$\frac{NP}{OP} = \sin AOP = \sin \theta,$$
$$\frac{\text{arc } AP}{OP} = \text{number of radians in } \angle AOP = \theta \text{ (Art. 158)},$$
and
$$\frac{PT}{OP} = \tan POT = \tan AOP = \tan \theta.$$

Hence $\sin\theta$, θ, and $\tan\theta$ are in ascending order of magnitude, provided that
$$\theta < \frac{\pi}{2}.$$

205. Since $\sin\theta < \theta < \tan\theta$, we have, by dividing each by the positive quantity $\sin\theta$,
$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Hence $\frac{\theta}{\sin\theta}$ always lies between 1 and $\frac{1}{\cos\theta}$.

This holds however small θ may be.

Now, when θ is very small, $\cos\theta$ is very nearly unity, and the smaller θ becomes, the more nearly does $\cos\theta$ become unity, and hence the more nearly does $\frac{1}{\cos\theta}$ become unity.

Hence, when θ is very small, the quantity $\frac{\theta}{\sin\theta}$ lies between 1 and a quantity which differs from unity by an indefinitely small quantity.

In other words, when θ is made indefinitely small the quantity $\frac{\theta}{\sin\theta}$, and therefore $\frac{\sin\theta}{\theta}$, is ultimately equal to unity, *i.e.* the smaller an angle becomes the more nearly is its sine equal to the number of radians in it.

This is often shortly expressed thus;

$$\sin\theta = \theta, \text{ when } \theta \text{ is very small.}$$

So also $\tan\theta = \theta$, when θ is very small.

206. In the preceding article it must be particularly noticed that θ is the number of radians in the angle considered.

The value of $\sin a^\circ$, when a is small, may be found. For, since $\pi^c = 180^\circ$, we have

$$a^\circ = \left(\pi \frac{a}{180}\right)^c.$$

$$\therefore \sin a^\circ = \sin\left(\frac{\pi a}{180}\right)^c = \frac{\pi a}{180},$$

by the result of the last article.

207. From the tables it will be seen that the sine of an angle and its circular measure agree to 7 places of decimals so long as the angle is not greater than 18′. They agree to the 5th place of decimals so long as the angle is less than about 2°.

208. **Ex. 1.** *Find the values of* $\sin 10'$ *and* $\cos 10'$.

Since $$10' = \frac{1^\circ}{6} = \frac{\pi^c}{180\times 6},$$

we have $$\sin 10' = \sin\left(\frac{\pi}{180\times 6}\right)^c = \frac{\pi}{180\times 6}$$

$$= \frac{3\cdot14159265\ldots}{180\times 6} = \cdot0029089 \text{ nearly.}$$

Also $$\cos 10' = \sqrt{1-\sin^2 10'}$$

$$= [1 - \cdot000008468\ldots]^{\frac{1}{2}}$$

$$= 1 - \frac{1}{2}[\cdot000008468\ldots],$$

approximately by the Binomial Theorem,

$$= 1 - \cdot000004234\ldots$$

$$= \cdot9999958\ldots.$$

Ex. 2. *Solve approximately the equation*

$$\sin\theta = \cdot 52.$$

Since $\sin\theta$ is very nearly equal to $\frac{1}{2}$, θ must be nearly equal to $\frac{\pi}{6}$.

Let then $\theta = \frac{\pi}{6} + x$, where x is small.

$$\therefore\ \cdot 52 = \sin\left(\frac{\pi}{6} + x\right) = \sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x$$

$$= \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x.$$

Since x is very small, we have

$$\cos x = 1 \text{ and } \sin x = x \text{ nearly.}$$

$$\therefore\ \cdot 52 = \frac{1}{2} + \frac{\sqrt{3}}{2}x.$$

$$\therefore\ x = \cdot 02 \times \frac{2}{\sqrt{3}} \text{ radians} = \frac{\sqrt{3}^c}{75} = \frac{\sqrt{3}}{75} \times \frac{180^\circ}{\pi} = \left(\tfrac{12}{5} \times 1\cdot 732 \ldots \times \cdot 318 \ldots\right)^\circ$$

$$= 1\cdot 32^\circ \text{ nearly} = 1^\circ\, 19' \text{ nearly.}$$

Hence $\qquad \theta = 31^\circ\, 19'$ nearly.

EXAMPLES. XLII.

$$\left[\pi = 3\cdot 14159265\,;\ \frac{1}{\pi} = \cdot 31831\ldots\right]$$

Find, to 5 places of decimals, the value of

1. $\sin 7'$. **2.** $\sin 15''$. **3.** $\sin 1'$.

4. $\cos 15'$. **5.** $\operatorname{cosec} 8''$. **6.** $\sec 5'$.

Solve approximately the equations

7. $\sin\theta = \cdot 01$. **8.** $\sin\theta = \cdot 48$.

9. $\cos\left(\frac{\pi}{3} + \theta\right) = \cdot 49$. **10.** $\cos\theta = \cdot 999$.

11. Find approximately the distance at which a halfpenny, which is an inch in diameter, must be placed so as to just hide the moon, the angular diameter of the moon, that is the angle its diameter subtends at the observer's eye, being taken to be $30'$.

12. A person walks in a straight line toward a very distant object, and observes that at three points A, B, and C the angles of elevation of the top of the object are α, 2α, and 3α respectively; prove that

$$AB = 3BC \text{ nearly.}$$

209. Area of a circle.

By Art. 142, the area of a regular polygon of n sides, which is inscribed in a circle of radius R,

$$=\frac{n}{2}R^2\sin\frac{360^\circ}{n}=\frac{n}{2}R^2\sin\frac{2\pi}{n}.$$

Let now the number of sides of this polygon be indefinitely increased, the polygon always remaining regular.

It is clear that the perimeter of the polygon must more and more approximate to the circumference of the circle.

Hence, when the number of sides of the polygon is infinitely great, the area of the circle must be the same as that of the polygon.

$$\text{Now}\quad \frac{n}{2}R^2\sin\frac{2\pi}{n}=\frac{n}{2}R^2.\frac{2\pi}{n}.\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}=\pi R^2.\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}$$

$$=\pi R^2.\frac{\sin\theta}{\theta},\ \text{where}\ \theta=\frac{2\pi}{n}.$$

When n is made infinitely great, the value of θ becomes infinitely small, and then, by Art. 205, $\frac{\sin\theta}{\theta}$ is unity.

The area of the circle therefore $=\boldsymbol{\pi}\mathbf{R}^2=\pi$ times the square of its radius.

210. *Area of the sector of a circle.*

Let O be the centre of a circle, AB the bounding arc of the sector, and let $\angle AOB=\alpha$ radians.

By geometry, since sectors are to one another as the arcs on which they stand, we have

$$\frac{\text{area of sector } AOB}{\text{area of whole circle}}=\frac{\text{arc } AB}{\text{circumference}}$$

$$=\frac{R\alpha}{2\pi R}=\frac{\alpha}{2\pi}.$$

$$\therefore \text{ area of sector } AOB = \frac{a}{2\pi} \times \text{area of whole circle}$$
$$= \frac{a}{2\pi} \times \pi R^2 = \frac{1}{2} R^2 . a.$$

EXAMPLES. XLIII.

[*In the following examples assume that* $\pi = 3{\cdot}14159...$; $\frac{1}{\pi} = {\cdot}31831$ *and* $\log \pi = {\cdot}49715$.]

1. Find the area of a circle whose circumference is 74 feet.

2. The diameter of a circle is 10 feet; find the area of a sector whose arc is $22\frac{1}{2}°$.

3. The area of a certain sector of a circle is 10 square feet; if the radius of the circle be 3 feet, find the angle of the sector.

4. The perimeter of a certain sector of a circle is 10 feet; if the radius of the circle be 3 feet, find the area of the sector.

5. What is the length of the circumference of a circle whose area is one acre?

6. A circle is of radius one foot; find (1) the area between it and an inscribed square, and (2) the area between it and a circumscribed square.

7. Find to the nearest hundredth of an inch the radius of a circle equal in area to an equilateral triangle of side 3 inches.

8. Find the ratio of the area of a regular polygon of 120 sides to that of its circumscribing circle.

9. A circle of 10 feet circumference is surrounded by a uniform width equal in area to the circle. Find the diameter of the outer circle.

10. A strip of paper, two miles long and ·003 of an inch thick, is rolled up into a solid cylinder; find approximately the radius of the circular ends of the cylinder.

11. A strip of paper, one mile long, is rolled tightly up into a solid cylinder, the diameter of whose circular ends is 6 inches; find the thickness of the paper.

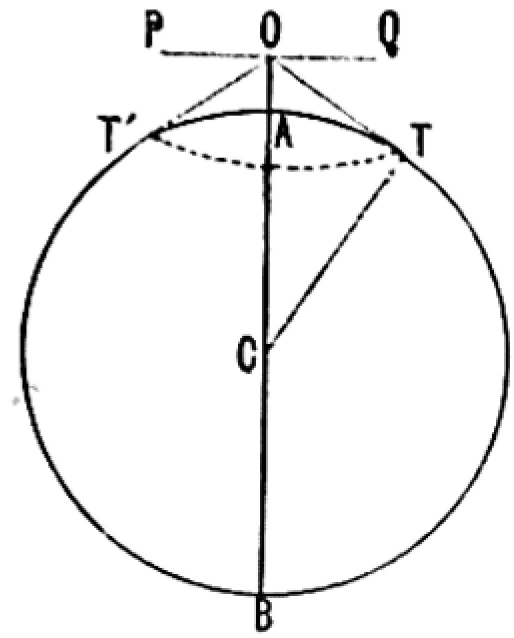
12. If each of three circles, of radius a, touch the other two, prove that the area included between them is nearly equal to $\frac{4}{25}a^2$.

13. Six equal circles, each of radius a, are placed so that each touches two others, their centres being all on the circumference of another circle; prove that the area which they enclose is

$$2a^2(3\sqrt{3} - \pi).$$

211. Dip of the Horizon.

Let O be a point at a distance h above the earth's surface. Draw tangents, such as OT and OT', to the surface of the earth. The ends of all these tangents all clearly lie on a circle. This circle is called the **Offing or Visible Horizon.** The angle that each of these tangents OT makes with a horizontal plane POQ is called the **Dip** of the Horizon at O.



Let r be the radius of the earth, and let B be the other end of the diameter through A.

Since $OC = OA + AC = h + r$, therefore

$$OT^2 = OC^2 - CT^2 = (r+h)^2 - r^2 = 2hr + h^2,$$

so that $$OT = \sqrt{h(2r+h)}.$$

This gives an accurate value for OT.

In all practical cases, however, h is very small compared with r.

[$r = 4000$ miles nearly, and h is never greater, and generally is very considerably less, than 5 miles.]

Hence h^2 is very small compared with hr.

As a close approximation, we have then

$$OT = \sqrt{2hr}.$$

The dip
$$= \angle TOQ$$
$$= 90^\circ - \angle COT = \angle OCT.$$

Also, $$\tan OCT = \frac{OT}{CT} = \frac{\sqrt{2hr}}{r} = \sqrt{\frac{2h}{r}},$$

so that, very approximately, we have

$$\angle OCT = \sqrt{\frac{2h}{r}} \text{ radians}$$

$$= \left(\sqrt{\frac{2h}{r}}\,\frac{180}{\pi}\right)^\circ = \left[\frac{180 \times 60 \times 60}{\pi}\sqrt{\frac{2h}{r}}\right]''.$$

212. **Ex.** *Taking the radius of the earth as* 4000 *miles, find the dip at the top of a lighthouse which is* 264 *feet above the sea, and the distance of the offing.*

Here $r = 4000$ miles, and $h = 264$ feet $= \frac{1}{20}$ mile.

Hence h is very small compared with r, so that

$$OT = \sqrt{\frac{1}{10} \times 4000} = \sqrt{400} = 20 \text{ miles.}$$

Also the dip $= \sqrt{\frac{2h}{r}}$ radians $= \frac{1}{200}$ radian

$$= \left(\frac{1}{200} \times \frac{180 \times 60}{\pi}\right)' = \left(\frac{54}{\pi}\right)' = 17'11'' \text{ nearly.}$$

EXAMPLES. XLIV.

[*Unless otherwise stated, the earth's radius may be taken to be* 4000 *miles.*]

1. Find in degrees, minutes, and seconds, the dip of the horizon from the top of a mountain 4200 feet high, the earth's radius being 21×10^6 feet.

2. The lamp of a lighthouse is 196 feet high; how far off can it be seen?

3. If the radius of the earth be 4000 miles, find the height of a balloon when the dip is 1°.

Find also the dip when the balloon is 2 miles high.

4. From the top of the mast of a ship, which is 66 feet above the sea, the light of a lighthouse which is known to be 132 feet high can just be seen; prove that its distance is 24 miles nearly.

5. From the top of a mast, 66 feet above the sea, the top of the mast of another ship can just be seen at a distance of 20 miles; prove that the heights of the masts are the same.

6. Prove that, if the height of the place of observation be n feet, the distance that the observer can see is $\sqrt{\frac{3n}{2}}$ miles nearly.

7. There are 10 million metres in a quadrant of the earth's circumference. Find approximately the distance at which the top of the Eiffel tower should be visible, its height being 300 metres.

TABLES OF LOGARITHMS, NATURAL SINES, NATURAL TANGENTS, LOGARITHMIC SINES, AND LOGARITHMIC TANGENTS.

TABLE I.

LOGARITHMS OF NUMBERS.

	0	1	2	3	4	5	6	7	8	9	Differences. 1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

LOGARITHMS OF NUMBERS.

	0	1	2	3	4	5	6	7	8	9	Differences. 1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

NATURAL SINES.

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Differences.								
0°	0·0000	0·0029	0·0058	0·0087	0·0116	0·0145	0·0175	89°	3	6	9	12	15	17	20	23	26
1	·0175	·0204	·0233	·0262	·0291	·0320	·0349	83	3	6	9	12	15	17	20	23	26
2	·0349	·0378	·0407	·0436	·0465	·0494	·0523	87	3	6	9	12	15	17	20	23	26
3	·0523	·0552	·0581	·0610	·0640	·0669	·0698	86	3	6	9	12	15	17	20	23	26
4	·0698	·0727	·0756	·0785	·0814	·0843	·0872	85	3	6	9	12	15	17	20	23	26
5	0·0872	0·0901	0·0929	0·0958	0·0987	0·1016	0·1045	84	3	6	9	12	14	17	20	23	26
6	·1045	·1074	·1103	·1132	·1161	·1190	·1219	83	3	6	9	12	14	17	20	23	26
7	·1219	·1248	·1276	·1305	·1334	·1363	·1392	82	3	6	9	12	14	17	20	23	26
8	·1392	·1421	·1449	·1478	·1507	·1536	·1564	81	3	6	9	11	14	17	20	23	26
9	·1564	·1593	·1622	·1650	·1679	·1708	·1736	80°	3	6	9	11	14	17	20	23	26
10°	0·1736	0·1765	0·1794	0·1822	0·1851	0·1880	0·1908	79	3	6	9	11	14	17	20	23	26
11	·1908	·1937	·1965	·1994	·2022	·2051	·2079	78	3	6	9	11	14	17	20	23	26
12	·2079	·2108	·2136	·2164	·2193	·2221	·2250	77	3	6	9	11	14	17	20	23	26
13	·2250	·2278	·2306	·2334	·2363	·2391	·2419	76	3	6	8	11	14	17	20	23	25
14	·2419	·2447	·2476	·2504	·2532	·2560	·2588	75	3	6	8	11	14	17	20	23	25
15	0·2588	0·2616	0·2644	0·2672	0·2700	0·2728	0·2756	74	3	6	8	11	14	17	20	22	25
16	·2756	·2784	·2812	·2840	·2868	·2896	·2924	73	3	6	8	11	14	17	20	22	25
17	·2924	·2952	·2979	·3007	·3035	·3062	·3090	72	3	6	8	11	14	17	19	22	25
18	·3090	·3118	·3145	·3173	·3201	·3228	·3256	71	3	6	8	11	14	17	19	22	25
19	·3256	·3283	·3311	·3338	·3365	·3393	·3420	70°	3	5	8	11	14	16	19	22	25

20°	0·3420	0·3448	0·3475	0·3502	0·3529	0·3557	0·3584	69	3	5	8	11	14	16	19	22	25
21	·3584	·3611	·3638	·3665	·3692	·3719	·3746	68	3	5	8	11	14	16	19	22	24
22	·3746	·3773	·3800	·3827	·3854	·3881	·3907	67	3	5	8	11	13	16	19	21	24
23	·3907	·3934	·3961	·3987	·4014	·4041	·4067	66	3	5	8	11	13	16	19	21	24
24	·4067	·4094	·4120	·4147	·4173	·4200	·4226	65	3	5	8	11	13	16	19	21	24
25	0·4226	0·4253	0·4279	0·4305	0·4331	0·4358	0·4384	64	3	5	8	10	13	16	18	21	24
26	·4384	·4410	·4436	·4462	·4488	·4514	·4540	63	3	5	8	10	13	16	18	21	23
27	·4540	·4566	·4592	·4617	·4643	·4669	·4695	62	3	5	8	10	13	15	18	21	23
28	·4695	·4720	·4746	·4772	·4797	·4823	·4848	61	3	5	8	10	13	15	18	20	23
29	·4848	·4874	·4899	·4924	·4950	·4975	·5000	60°	3	5	8	10	13	15	18	20	23
30°	0·5000	0·5025	0·5050	0·5075	0·5100	0·5125	0·5150	59	3	5	8	10	13	15	18	20	23
31	·5150	·5175	·5200	·5225	·5250	·5275	·5299	58	2	5	7	10	12	15	17	20	22
32	·5299	·5324	·5348	·5373	·5398	·5422	·5446	57	2	5	7	10	12	15	17	20	22
33	·5446	·5471	·5495	·5519	·5544	·5568	·5592	56	2	5	7	10	12	15	17	19	22
34	·5592	·5616	·5640	·5664	·5688	·5712	·5736	55	2	5	7	10	12	14	17	19	22
35	0·5736	0·5760	0·5783	0·5807	0·5831	0·5854	0·5878	54	2	5	7	9	12	14	17	19	21
36	·5878	·5901	·5925	·5948	·5972	·5995	·6018	53	2	5	7	9	12	14	16	19	21
37	·6018	·6041	·6065	·6088	·6111	·6134	·6157	52	2	5	7	9	12	14	16	18	21
38	·6157	·6180	·6202	·6225	·6248	·6271	·6293	51	2	5	7	9	11	14	16	18	20
39	·6293	·6316	·6338	·6361	·6383	·6406	·6428	50°	2	4	7	9	11	13	16	18	20
40°	0·6428	0·6450	0·6472	0·6494	0·6517	0·6539	0·6561	49	2	4	7	9	11	13	15	18	20
41	·6561	·6583	·6604	·6626	·6648	·6670	·6691	48	2	4	7	9	11	13	15	17	20
42	·6691	·6713	·6734	·6756	·6777	·6799	·6820	47	2	4	6	9	11	13	15	17	19
43	·6820	·6841	·6862	·6884	·6905	·6926	·6947	46	2	4	6	8	11	13	15	17	19
44	·6947	·6967	·6988	·7009	·7030	·7050	·7071	45	2	4	6	8	10	12	15	17	19
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

NATURAL COSINES.

NATURAL SINES.

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Differences.								
45°	0·7071	0·7092	0·7112	0·7133	0·7153	0·7173	0·7193	44°	2	4	6	8	10	12	14	16	18
46	·7193	·7214	·7234	·7254	·7274	·7294	·7314	43	2	4	6	8	10	12	14	16	18
47	·7314	·7333	·7353	·7373	·7392	·7412	·7431	42	2	4	6	8	10	12	14	16	18
48	·7431	·7451	·7470	·7490	·7509	·7528	·7547	41	2	4	6	8	10	12	13	15	17
49	·7547	·7566	·7585	·7604	·7623	·7642	·7660	40°	2	4	6	8	9	11	13	15	17
50°	0·7660	0·7679	0·7698	0·7716	0·7735	0·7753	0·7771	39	2	4	6	7	9	11	13	15	17
51	·7771	·7790	·7808	·7826	·7844	·7862	·7880	38	2	4	5	7	9	11	13	14	16
52	·7880	·7898	·7916	·7934	·7951	·7969	·7986	37	2	4	5	7	9	11	12	14	16
53	·7986	·8004	·8021	·8039	·8056	·8073	·8090	36	2	3	5	7	9	10	12	14	16
54	·8090	·8107	·8124	·8141	·8158	·8175	·8192	35	2	3	5	7	8	10	12	14	15
55	0·8192	0·8208	0·8225	0·8241	0·8258	0·8274	0·8290	34	2	3	5	7	8	10	12	13	15
56	·8290	·8307	·8323	·8339	·8355	·8371	·8387	33	2	3	5	6	8	10	11	13	14
57	·8387	·8403	·8418	·8434	·8450	·8465	·8480	32	2	3	5	6	8	9	11	13	14
58	·8480	·8496	·8511	·8526	·8542	·8557	·8572	31	2	3	5	6	8	9	11	12	14
59	·8572	·8587	·8601	·8616	·8631	·8646	·8660	30°	1	3	4	6	7	9	10	12	13
60°	0·8660	0·8675	0·8689	0·8704	0·8718	0·8732	0·8746	29	1	3	4	6	7	9	10	11	13
61	·8746	·8760	·8774	·8788	·8802	·8816	·8829	28	1	3	4	6	7	8	10	11	12
62	·8829	·8843	·8857	·8870	·8884	·8897	·8910	27	1	3	4	5	7	8	9	11	12
63	·8910	·8923	·8936	·8949	·8962	·8975	·8988	26	1	3	4	5	6	8	9	10	12
64	·8988	·9001	·9013	·9026	·9038	·9051	·9063	25	1	3	4	5	6	8	9	10	11

65	0·9063	0·9075	0·9088	0·9100	0·9112	0·9124	0·9135	24	1	2	4	5	6	7	8	10	11
66	·9135	·9147	·9159	·9171	·9182	·9194	·9205	23	1	2	3	5	6	7	8	9	10
67	·9205	·9216	·9228	·9239	·9250	·9261	·9272	22	1	2	3	4	6	7	8	9	10
68	·9272	·9283	·9293	·9304	·9315	·9325	·9336	21	1	2	3	4	5	6	7	9	10
69	·9336	·9346	·9356	·9367	·9377	·9387	·9397	20°	1	2	3	4	5	6	7	8	9
70°	0·9397	0·9407	0·9417	0·9426	0·9436	0·9446	0·9455	19	1	2	3	4	5	6	7	8	9
71	·9455	·9465	·9474	·9483	·9492	·9502	·9511	18	1	2	3	4	5	6	6	7	8
72	·9511	·9520	·9528	·9537	·9546	·9555	·9563	17	1	2	3	4	4	5	6	7	8
73	·9563	·9572	·9580	·9588	·9596	·9605	·9613	16	1	2	2	3	4	5	6	7	7
74	·9613	·9621	·9628	·9636	·9644	·9652	·9659	15	1	2	2	3	4	5	5	6	7
75	0·9659	0·9667	0·9674	0·9681	0·9689	0·9696	0·9703	14	1	1	2	3	4	4	5	6	7
76	·9703	·9710	·9717	·9724	·9730	·9737	·9744	13	1	1	2	3	3	4	5	5	6
77	·9744	·9750	·9757	·9763	·9769	·9775	·9781	12	1	1	2	3	3	4	4	5	6
78	·9781	·9787	·9793	·9799	·9805	·9811	·9816	11	1	1	2	2	3	3	4	5	5
79	·9816	·9822	·9827	·9833	·9838	·9843	·9848	10°	1	1	2	2	3	3	4	4	5
80°	0·9848	0·9853	0·9858	0·9863	0·9868	0·9872	0·9877	9	0	1	1	2	2	3	3	4	4
81	·9877	·9881	·9886	·9890	·9894	·9899	·9903	8	0	1	1	2	2	3	3	3	4
82	·9903	·9907	·9911	·9914	·9918	·9922	·9925	7	0	1	1	2	2	2	3	3	3
83	·9925	·9929	·9932	·9936	·9939	·9942	·9945	6	0	1	1	1	2	2	2	3	3
84	·9945	·9948	·9951	·9954	·9957	·9959	·9962	5	0	1	1	1	1	2	2	2	3
85	0·9962	0·9964	0·9967	0·9969	0·9971	0·9974	0·9976	4	Differences are so small here that tabulation is unnecessary.								
86	·9976	·9978	·9980	·9981	·9983	·9985	·9986	3									
87	·9986	·9988	·9989	·9990	·9992	·9993	·9994	2									
88	·9994	·9995	·9996	·9997	·9997	·9998	·9998	1									
89	0·9998	0·9999	0·9999	1·0000	1·0000	1·0000	1·0000	0°									
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

NATURAL COSINES.

NATURAL TANGENTS.

	0′	10′	20′	30′	40′	50′	60′		Differences. 1′	2′	3′	4′	5′	6′	7′	8′	9′
0°	0·0000	0·0029	0·0058	0·0087	0·0116	0·0145	0·0175	89°	3	6	9	12	15	17	20	23	26
1	·0175	·0204	·0233	·0262	·0291	·0320	·0349	88	3	6	9	12	15	17	20	23	26
2	·0349	·0378	·0407	·0437	·0466	·0495	·0524	87	3	6	9	12	15	18	20	23	26
3	·0524	·0553	·0582	·0612	·0641	·0670	·0699	86	3	6	9	12	15	18	20	23	26
4	·0699	·0729	·0758	·0787	·0816	·0846	·0875	85	3	6	9	12	15	18	21	23	26
5	0·0875	0·0904	0·0934	0.0963	0·0992	0·1022	0·1051	84	3	6	9	12	15	18	21	24	26
6	·1051	·1080	·1110	·1139	·1169	·1198	·1228	83	3	6	9	12	15	18	21	24	27
7	·1228	·1257	·1287	·1317	·1346	·1376	·1405	82	3	6	9	12	15	18	21	24	27
8	·1405	·1435	·1465	·1495	·1524	·1554	·1584	81	3	6	9	12	15	18	21	24	27
9	·1584	·1614	·1644	·1673	·1703	·1733	·1763	80°	3	6	9	12	15	18	21	24	27
10°	0·1763	0·1793	0·1823	0·1853	0·1883	0·1914	0·1944	79	3	6	9	12	15	18	21	24	27
11	·1944	·1974	·2004	·2035	·2065	·2095	·2126	78	3	6	9	12	15	18	21	24	27
12	·2126	·2156	·2186	·2217	·2247	·2278	·2309	77	3	6	9	12	15	18	21	24	27
13	·2309	·2339	·2370	·2401	·2432	·2462	·2493	76	3	6	9	12	15	18	22	25	28
14	·2493	·2524	·2555	·2586	·2617	·2648	·2679	75	3	6	9	12	16	19	22	25	28
15	0·2679	0·2711	0·2742	0·2773	0·2805	0·2836	0·2867	74	3	6	9	13	16	19	22	25	28
16	·2867	·2899	·2931	·2962	·2994	·3026	·3057	73	3	6	9	13	16	19	22	25	28
17	·3057	·3089	·3121	·3153	·3185	·3217	·3249	72	3	6	10	13	16	19	22	26	29
18	·3249	·3281	·3314	·3346	·3378	·3411	·3443	71	3	6	10	13	16	19	23	26	29
19	·3443	·3476	·3508	·3541	·3574	·3607	·3640	70°	3	7	10	13	16	20	23	26	29

20°	0·3640	0·3673	0·3706	0·3739	0·3772	0·3805	0·3839	69	3	7	10	13	17	20	23	27	30
21	·3839	·3872	·3906	·3939	·3973	·4006	·4040	68	3	7	10	13	17	20	24	27	30
22	·4040	·4074	·4108	·4142	·4176	·4210	·4245	67	3	7	10	14	17	20	24	27	31
23	·4245	·4279	·4314	·4348	·4383	·4417	·4452	66	3	7	10	14	17	21	24	28	31
24	·4452	·4487	·4522	·4557	·4592	·4628	·4663	65	4	7	11	14	18	21	25	28	32
25	0·4663	0·4699	0.4734	0·4770	0·4806	0·4841	0·4877	64	4	7	11	14	18	21	25	29	32
26	·4877	·4913	·4950	·4986	·5022	·5059	·5095	63	4	7	11	15	18	22	25	29	33
27	·5095	·5132	·5169	·5206	·5243	·5280	·5317	62	4	7	11	15	18	22	26	30	33
28	·5317	·5354	·5392	·5430	·5467	·5505	·5543	61	4	8	11	15	19	23	26	30	34
29	·5543	·5581	·5619	·5658	·5696	·5735	·5774	60°	4	8	12	15	19	23	27	31	35
30°	0·5774	0·5812	0·5851	0·5890	0·5930	0·5969	0·6009	59	4	8	12	16	20	24	27	31	35
31	·6009	·6048	·6088	·6128	·6168	·6208	·6249	58	4	8	12	16	20	24	28	32	36
32	·6249	·6289	·6330	·6371	·6412	·6453	·6494	57	4	8	12	16	20	25	29	33	37
33	·6494	·6536	·6577	·6619	·6661	·6703	·6745	56	4	8	13	17	21	25	29	33	38
34	·6745	·6787	·6830	·6873	·6916	·6959	·7002	55	4	9	13	17	21	26	30	34	39
35	0·7002	0·7046	0·7089	0·7133	0·7177	0·7221	0·7265	54	4	9	13	18	22	26	31	35	40
36	·7265	·7310	·7355	·7400	·7445	·7490	·7536	53	5	9	14	18	23	27	32	36	41
37	·7536	·7581	·7627	·7673	·7720	·7766	·7813	52	5	9	14	18	23	28	32	37	42
38	·7813	·7860	·7907	·7954	·8002	·8050	·8098	51	5	10	14	19	24	29	33	38	43
39	·8098	·8146	·8195	·8243	·8292	·8342	·8391	50°	5	10	15	20	24	29	34	39	44
40°	0·8391	0·8441	0·8491	0·8541	0·8591	0·8642	0·8693	49	5	10	15	20	25	30	35	40	45
41	·8693	·8744	·8796	·8847	·8899	·8952	·9004	48	5	10	16	21	26	31	36	41	47
42	·9004	·9057	·9110	·9163	·9217	·9271	·9325	47	5	11	16	21	27	32	37	43	48
43	·9325	·9380	·9435	·9490	·9545	·9601	·9657	46	6	11	17	22	28	33	39	44	50
44	·9657	·9713	·9770	·9827	·9884	·9942	1·0000	45	6	11	17	23	29	34	40	46	51
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

NATURAL COTANGENTS.

NATURAL TANGENTS.

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Differences.								
45°	1·0000	1·0058	1·0117	1·0176	1·0235	1·0295	1·0355	44°	6	12	18	24	30	36	41	47	53
46	·0355	·0416	·0477	·0538	·0599	·0661	·0724	43	6	12	18	25	31	37	43	49	55
47	·0724	·0786	·0850	·0913	·0977	·1041	·1106	42	6	13	19	26	32	38	45	51	57
48	·1106	·1171	·1237	·1303	·1369	·1436	·1504	41	7	13	20	27	33	40	46	53	60
49	·1504	·1571	·1640	·1708	·1778	·1847	·1918	40°	7	14	21	28	34	41	48	55	62
50°	1·1918	1·1988	1·2059	1·2131	1·2203	1·2276	1·2349	39	7	14	22	29	36	43	50	58	65
51	·2349	·2423	·2497	·2572	·2647	·2723	·2799	38	8	15	23	30	38	45	53	60	68
52	·2799	·2876	·2954	·3032	·3111	·3190	·3270	37	8	16	24	31	39	47	55	63	71
53	·3270	·3351	·3432	·3514	·3597	·3680	·3764	36	8	16	25	33	41	49	58	66	74
54	·3764	·3848	·3934	·4019	·4106	·4193	·4281	35	9	17	26	35	43	52	60	69	78
55	1·4281	1·4370	1·4460	1·4550	1·4641	1·4733	1·4826	34	9	18	27	36	45	54	63	73	82
56	·4826	·4919	·5013	·5108	·5204	·5301	·5399	33	10	19	29	38	48	57	67	76	86
57	·5399	·5497	·5597	·5697	·5798	·5900	·6003	32	10	20	30	40	50	60	71	81	91
58	·6003	·6107	·6212	·6319	·6426	·6534	·6643	31	11	21	32	43	53	64	75	85	96
59	·6643	·6753	·6864	·6977	·7090	·7205	·7321	30°	11	23	34	45	56	68	79	90	102
60°	1·732	1·744	1·756	1·767	1·780	1·792	1·804	29	1	2	4	5	6	7	8	10	11
61	1·804	1·816	1·829	1·842	1·855	1·868	1·881	28	1	3	4	5	6	8	9	10	12
62	1·881	1·894	1·907	1·921	1·935	1·949	1·963	27	1	3	4	5	7	8	10	11	12
63	1·963	1·977	1·991	2·006	2·020	2·035	2·050	26	1	3	4	6	7	9	10	12	13
64	2·050	2·066	2·081	2·097	2·112	2·128	2·145	25	2	3	5	6	8	9	11	13	14

65	2·145	2·161	2·177	2·194	2·211	2·229	2·246	24	2	3	5	7	8	10	12	14	15
66	2·246	2·264	2·282	2·300	2·318	2·337	2·356	23	2	4	5	7	9	11	13	15	16
67	2·356	2·375	2·394	2·414	2·434	2·455	2·475	22	2	4	6	8	10	12	14	16	18
68	2·475	2·496	2·517	2·539	2·560	2·583	2·605	21	2	4	6	9	11	13	15	17	20
69	2·605	2·628	2·651	2·675	2·699	2·723	2·747	20°	2	5	7	9	12	14	17	19	21
70°	2·747	2·773	2·798	2·824	2·850	2·877	2·904	19	3	5	8	10	13	16	18	21	23
71	2·904	2·932	2·960	2·989	3·018	3·047	3·078	18	3	6	9	12	14	17	20	23	26
72	3·078	3·108	3·140	3·172	3·204	3·237	3·271	17	3	6	10	13	16	19	23	26	29
73	3·271	3·305	3·340	3·376	3·412	3·450	3·487	16	4	7	11	14	18	22	25	29	32
74	3·487	3·526	3·566	3·606	3·647	3·689	3·732	15	4	8	12	16	20	24	29	33	37
75	3·732	3·776	3·821	3·867	3·914	3·962	4·011	14	5	9	14	19	23	28	33	37	42
76	4·011	4·061	4·113	4·165	4·219	4·275	4·331	13	5	11	16	21	27	32	37	43	48
77	4·331	4·390	4·449	4·511	4·574	4·638	4·705	12	6	12	19	25	31	37	44	50	56
78	4·705	4·773	4·843	4·915	4·989	5·066	5·145	11	7	15	22	29	37	44	51	59	66
79	5·145	5·226	5·309	5·396	5·485	5·576	5·671	10°	9	18	26	35	44	53	61	70	79
80°	5·671	5·769	5·871	5·976	6·084	6·197	6·314	9	The differences change so rapidly here that they cannot be tabulated.								
81	6·314	6·435	6·561	6·691	6·827	6·968	7·115	8									
82	7·115	7·269	7·429	7·596	7·770	7·953	8·144	7									
83	8·144	8·345	8·556	8·777	9·010	9·255	9·514	6									
84	9·514	9·788	10·078	10·385	10·712	11·059	11·430	5									
85	11·43	11·83	12·25	12·71	13·20	13·73	14·30	4	The cotangent of a small angle of *n* minutes of arc or the tangent of 90° minus *n* minutes is very nearly equal to 3438 divided by *n*.								
86	14·30	14·92	15·60	16·35	17·17	18·07	19·08	3									
87	19·08	20·21	21·47	22·90	24·54	26·43	28·64	2									
88	28·64	31·24	34·37	38·19	42·96	49·10	57·29	1									
89	57·29	68·75	85·94	114·59	171·89	343·77	∞	0°									
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

NATURAL COTANGENTS.

LOGARITHMIC SINES.

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Differences.								
0°	$-\infty$	7·4637	7·7648	7·9408	8·0658	8·1627	8·2419	89°	For small angles of n minutes of arc, log sine n' or log cosine $(90° - n')$ $= \log n + \bar{4}{\cdot}4637$ Differences vary so rapidly here that tabulation is impossible.								
1	8·2419	8·3088	8·3668	8·4179	8·4637	8·5050	8·5428	88									
2	8·5428	8·5776	8·6097	8·6397	8·6677	8·6940	8·7188	87									
3	8·7188	8·7423	8·7645	8·7857	8·8059	8·8251	8·8436	86									
4	8·8436	8·8613	8·8783	8·8946	8·9104	8·9256	8·9403	85									
5	8·9403	8·9545	8·9682	8·9816	8·9945	9·0070	9·0192	84									
6	9·0192	9·0311	9·0426	9·0539	9·0648	9·0755	9·0859	83									
7	9·0859	9·0961	9·1060	9·1157	9·1252	9·1345	9·1436	82	10	19	29	39	48	58	67	77	87
8	9·1436	9·1525	9·1612	9·1697	9·1781	9·1863	9·1943	81	8	17	25	34	42	51	59	68	76
9	9·1943	9·2022	9·2100	9·2176	9·2251	9·2324	9·2397	80°	8	15	23	30	38	45	53	61	68
10°	9·2397	9·2468	9.2538	9·2606	9·2674	9·2740	9·2806	79	7	14	20	27	34	41	48	55	62
11	·2806	·2870	·2934	·2997	·3058	·3119	·3179	78	6	12	19	25	31	37	44	50	56
12	·3179	·3238	·3296	·3353	·3410	·3466	·3521	77	6	11	17	23	29	34	40	46	51
13	·3521	·3575	·3629	·3682	·3734	·3786	·3837	76	5	11	16	21	26	32	37	42	47
14	·3837	·3887	·3937	·3986	·4035	·4083	·4130	75	5	10	15	20	24	29	34	39	44
15	9·4130	9·4177	9·4223	9·4269	9·4314	9·4359	9·4403	74	5	9	14	18	23	27	32	36	41
16	·4403	·4447	·4491	·4533	·4576	·4618	·4659	73	4	9	13	17	21	26	30	34	38
17	·4659	·4700	·4741	·4781	·4821	·4861	·4900	72	4	8	12	16	20	24	28	32	36
18	·4900	·4939	·4977	·5015	·5052	·5090	·5126	71	4	8	11	15	19	23	26	30	34
19	·5126	·5163	·5199	·5235	·5270	·5306	·5341	70°	4	7	11	14	18	21	25	29	32

20°	9·5341	9·5375	9·5409	9·5443	9·5477	9·5510	9·5543	69	3	7	10	14	17	20	24	27	30
21	·5543	·5576	·5609	·5641	·5673	·5704	·5736	68	3	6	10	13	16	19	22	26	29
22	·5736	·5767	·5798	·5828	·5859	·5889	·5919	67	3	6	9	12	15	18	21	24	27
23	·5919	·5948	·5978	·6007	·6036	·6065	·6093	66	3	6	9	12	15	17	20	23	26
24	·6093	·6121	·6149	·6177	·6205	·6232	·6259	65	3	6	8	11	14	17	19	22	25
25	9·6259	9·6286	9·6313	9·6340	9·6366	9·6392	9·6418	64	3	5	8	11	13	16	19	21	24
26	·6418	·6444	·6470	·6495	·6521	·6546	·6570	63	3	5	8	10	13	15	18	20	23
27	·6570	·6595	·6620	·6644	·6668	·6692	·6716	62	2	5	7	10	12	15	17	19	22
28	·6716	·6740	·6763	·6787	·6810	·6833	·6856	61	2	5	7	9	12	14	16	19	21
29	·6856	·6878	·6901	·6923	·6946	·6968	·6990	60°	2	4	7	9	11	13	16	18	20
30°	9·6990	9·7012	9·7033	9·7055	9·7076	9·7097	9·7118	59	2	4	6	9	11	13	15	17	19
31	·7118	·7139	·7160	·7181	·7201	·7222	·7242	58	2	4	6	8	10	12	14	16	19
32	·7242	·7262	·7282	·7302	·7322	·7342	·7361	57	2	4	6	8	10	12	14	16	18
33	·7361	·7380	·7400	·7419	·7438	·7457	·7476	56	2	4	6	8	10	11	13	15	17
34	·7476	·7494	·7513	·7531	·7550	·7568	·7586	55	2	4	6	7	9	11	13	15	17
35	9·7586	9·7604	9·7622	9·7640	9·7657	9·7675	9·7692	54	2	4	5	7	9	11	12	14	16
36	·7692	·7710	·7727	·7744	·7761	·7778	·7795	53	2	3	5	7	9	10	12	14	15
37	·7795	·7811	·7828	·7844	·7861	·7877	·7893	52	2	3	5	7	8	10	12	13	15
38	·7893	·7910	·7926	·7941	·7957	·7973	·7989	51	2	3	5	6	8	10	11	13	14
39	·7989	·8004	·8020	·8035	·8050	·8066	·8081	50°	2	3	5	6	8	9	11	12	14
40°	9·8081	9·8096	9·8111	9·8125	9·8140	9·8155	9·8169	49	1	3	4	6	7	9	10	12	13
41	·8169	·8184	·8198	·8213	·8227	·8241	·8255	48	1	3	4	6	7	9	10	11	13
42	·8255	·8269	·8283	·8297	·8311	·8324	·8338	47	1	3	4	6	7	8	10	11	12
43	·8338	·8351	·8365	·8378	·8391	·8405	·8418	46	1	3	4	5	7	8	9	11	12
44	·8418	·8431	·8444	·8457	·8469	·8482	·8495	45	1	3	4	5	6	8	9	10	12
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

LOGARITHMIC COSINES.

LOGARITHMIC SINES.

	0′	10′	20′	30′	40	50′	60′		Differences. 1′	2′	3′	4′	5′	6′	7′	8′	9′
45°	9·8495	9·8507	9·8520	9·8532	9·8545	9·8557	9·8569	44°	1	2	4	5	6	7	9	10	11
46	·8569	·8582	·8594	·8606	·8618	·8629	·8641	43	1	2	4	5	6	7	8	10	11
47	·8641	·8653	·8665	·8676	·8688	·8699	·8711	42	1	2	3	5	6	7	8	9	10
48	·8711	·8722	·8733	·8745	·8756	·8767	·8778	41	1	2	3	4	6	7	8	9	10
49	·8778	·8789	·8800	·8810	·8821	·8832	·8843	40°	1	2	3	4	5	6	8	9	10
50°	9·8843	9·8853	9·8864	9·8874	9·8884	9·8895	9·8905	39	1	2	3	4	5	6	7	8	9
51	·8905	·8915	·8925	·8935	·8945	·8955	·8965	38	1	2	3	4	5	6	7	8	9
52	·8965	·8975	·8985	·8995	·9004	·9014	·9023	37	1	2	3	4	5	6	7	8	9
53	·9023	·9033	·9042	·9052	·9061	·9070	·9080	36	1	2	3	4	5	6	7	7	8
54	·9080	·9089	·9098	·9107	·9116	·9125	·9134	35	1	2	3	4	5	5	6	7	8
55	9·9134	9·9142	9·9151	9·9160	9·9169	9.9177	9·9186	34	1	2	3	3	4	5	6	7	8
56	·9186	·9194	·9203	·9211	·9219	·9228	·9236	33	1	2	3	3	4	5	6	7	8
57	·9236	·9244	·9252	·9260	·9268	·9276	·9284	32	1	2	2	3	4	5	6	6	7
58	·9284	·9292	·9300	·9308	·9315	·9323	·9331	31	1	2	2	3	4	5	5	6	7
59	·9331	·9338	·9346	·9353	·9361	·9368	·9375	30°	1	1	2	3	4	4	5	6	7
60°	9·9375	9·9383	9·9390	9·9397	9·9404	9·9411	9·9418	29	1	1	2	3	4	4	5	6	6
61	·9418	·9425	·9432	·9439	·9446	·9453	·9459	28	1	1	2	3	3	4	5	5	6
62	·9459	·9466	·9473	·9479	·9486	·9492	·9499	27	1	1	2	3	3	4	5	5	6
63	·9499	·9505	·9512	·9518	·9524	·9530	·9537	26	1	1	2	3	3	4	4	5	6
64	·9537	·9543	·9549	·9555	·9561	·9567	·9573	25	1	1	2	2	3	4	4	5	5

65	9·9573	9·9579	9·9584	9·9590	9·9596	9·9602	9·9607	24	1	1	2	2	3	3	4	5	5
66	·9607	·9613	·9618	·9624	·9629	·9635	·9640	23	1	1	2	2	3	3	4	4	5
67	·9640	·9646	·9651	·9656	·9661	·9667	·9672	22	1	1	2	2	3	3	4	4	5
68	·9672	·9677	·9682	·9687	·9692	·9697	·9702	21	0	1	1	2	2	3	3	4	4
69	·9702	·9706	·9711	·9716	·9721	·9725	·9730	20°	0	1	1	2	2	3	3	4	4
70°	9·9730	9·9734	9·9739	9·9743	9·9748	9·9752	9·9757	19	0	1	1	2	2	3	3	4	4
71	·9757	·9761	·9765	·9770	·9774	·9778	·9782	18	0	1	1	2	2	3	3	3	4
72	·9782	·9786	·9790	·9794	·9798	·9802	·9806	17	0	1	1	2	2	2	3	3	4
73	·9806	·9810	·9814	·9817	·9821	·9825	·9828	16	0	1	1	1	2	2	3	3	3
74	·9828	·9832	·9836	·9839	·9843	·9846	·9849	15	0	1	1	1	2	2	2	3	3
75	9·9849	9·9853	9·9856	9·9859	9·9863	9·9866	9·9869	14									
76	·9869	·9872	·9875	·9878	·9881	·9884	·9887	13									
77	·9887	·9890	·9893	·9896	·9899	·9901	·9904	12									
78	·9904	·9907	·9909	·9912	·9914	·9917	·9919	11									
79	·9919	·9922	·9924	·9927	·9929	·9931	·9934	10°	Differences are so small here that tabulation is unnecessary.								
80°	9·9934	9·9936	9·9938	9·9940	9·9942	9·9944	9·9946	9									
81	·9946	·9948	·9950	·9952	·9954	·9956	·9958	8									
82	·9958	·9959	·9961	·9963	·9964	·9966	·9968	7									
83	·9968	·9969	·9971	·9972	·9973	·9975	·9976	6									
84	·9976	·9977	·9979	·9980	·9981	·9982	·9983	5									
85	9·9983	9·9985	9·9986	9·9987	9·9988	9·9989	9·9989	4									
86	·9989	·9990	·9991	·9992	·9993	·9993	·9994	3									
87	·9994	·9995	·9995	·9996	·9996	·9997	·9997	2									
88	·9997	·9998	·9998	·9999	·9999	·9999	·9999	1									
89	9·9999	10·0000	10·0000	10·0000	10·0000	10·0000	10·0000	0°									
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

LOGARITHMIC COSINES.

LOGARITHMIC TANGENTS.

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
0°	$-\infty$	7·4637	7·7648	7·9409	8·0658	8·1627	8·2419	89°									
1	8·2419	8·3089	8·3669	8·4181	8·4638	8·5053	8·5431	88									
2	8·5431	8·5779	8·6101	8·6401	8·6682	8·6945	8·7194	87									
3	8·7194	8·7429	8·7652	8·7865	8·8067	8·8261	8·8446	86									
4	8·8446	8·8624	8·8795	8·8960	8·9118	8·9272	8·9420	85									
5	8·9420	8·9563	8·9701	8·9836	8·9966	9·0093	9·0216	84									
6	9·0216	9·0336	9·0453	9·0567	9·0678	9·0786	9·0891	83									
7	9·0891	9·0995	9·1096	9·1194	9·1291	9·1385	9·1478	82	10	20	29	39	49	59	69	78	88
8	9·1478	9·1569	9·1658	9·1745	9·1831	9·1915	9·1997	81	9	17	26	35	43	52	61	69	78
9	9·1997	9·2078	9·2158	9·2236	9·2313	9·2389	9·2463	80°	8	16	23	31	39	47	54	62	70
10°	9·2463	9·2536	9·2609	9·2680	9·2750	9·2819	9·2887	79	7	14	21	28	35	42	49	57	64
11	·2887	·2953	·3020	·3085	·3149	·3212	·3275	78	6	13	19	26	32	39	45	52	58
12	·3275	·3336	·3397	·3458	·3517	·3576	·3634	77	6	12	18	24	30	36	42	48	54
13	·3634	·3691	·3748	·3804	·3859	·3914	·3968	76	6	11	17	22	28	33	39	45	50
14	·3968	·4021	·4074	·4127	·4178	·4230	·4281	75	5	10	16	21	26	31	37	42	47
15	9·4281	9·4331	9·4381	9·4430	9·4479	9·4527	9·4575	74	5	10	15	20	25	30	34	39	44
16	·4575	·4622	·4669	·4716	·4762	·4808	·4853	73	5	9	14	19	23	28	33	37	42
17	·4853	·4898	·4943	·4987	·5031	·5075	·5118	72	4	9	13	18	22	26	31	35	40
18	·5118	·5161	·5203	·5245	·5287	·5329	·5370	71	4	8	13	17	21	25	29	34	38
19	·5370	·5411	·5451	·5491	·5531	·5571	·5611	70°	4	8	12	16	20	24	28	32	36

(Differences.) For small angles of n minutes of arc, log tan n'
or log cot $(90° - n')$
$= \log n + \bar{4}\cdot4637$
Differences vary so rapidly here that tabulation is impossible.

20°	9·5611	9·5650	9·5689	9·5727	9·5766	9·5804	9·5842	69	4	8	12	15	19	23	27	31	35
21	·5842	·5879	·5917	·5954	·5991	·6028	·6064	68	4	7	11	15	19	22	26	30	33
22	·6064	·6100	·6136	·6172	·6208	·6243	·6279	67	4	7	11	14	18	21	25	29	32
23	·6279	·6314	·6348	·6383	·6417	·6452	·6486	66	3	7	10	14	17	21	24	28	31
24	·6486	·6520	·6553	·6587	·6620	·6654	·6687	65	3	7	10	13	17	20	23	27	30
25	9·6687	9·6720	9·6752	9·6785	9·6817	9·6850	9·6882	64	3	7	10	13	16	20	23	26	29
26	·6882	·6914	·6946	·6977	·7009	·7040	·7072	63	3	6	9	13	16	19	22	25	28
27	·7072	·7103	·7134	·7165	·7196	·7226	·7257	62	3	6	9	12	15	19	22	25	28
28	·7257	·7287	·7317	·7348	·7378	·7408	·7438	61	3	6	9	12	15	18	21	24	27
29	·7438	·7467	·7497	·7526	·7556	·7585	·7614	60°	3	6	9	12	15	18	21	24	27
30°	9·7614	9·7644	9·7673	9·7701	9·7730	9·7759	9·7788	59	3	6	9	12	14	17	20	23	26
31	·7788	·7816	·7845	·7873	·7902	·7930	·7958	58	3	6	9	11	14	17	20	23	26
32	·7958	·7986	·8014	·8042	·8070	·8097	·8125	57	3	6	8	11	14	17	20	22	25
33	·8125	·8153	·8180	·8208	·8235	·8263	·8290	56	3	5	8	11	14	16	19	22	25
34	·8290	·8317	·8344	·8371	·8398	·8425	·8452	55	3	5	8	11	14	16	19	22	24
35	9·8452	9·8479	9·8506	9·8533	9·8559	9·8586	9·8613	54	3	5	8	11	13	16	19	21	24
36	·8613	·8639	·8666	·8692	·8718	·8745	·8771	53	3	5	8	11	13	16	19	21	24
37	·8771	·8797	·8824	·8850	·8876	·8902	·8928	52	3	5	8	10	13	16	18	21	24
38	·8928	·8954	·8980	·9006	·9032	·9058	·9084	51	3	5	8	10	13	16	18	21	23
39	·9084	·9110	·9135	·9161	·9187	·9212	·9238	50°	3	5	8	10	13	15	18	21	23
40°	9·9238	9·9264	9·9289	9·9315	9·9341	9·9366	9·9392	49	3	5	8	10	13	15	18	20	23
41	·9392	·9417	·9443	·9468	·9494	·9519	·9544	48	3	5	8	10	13	15	18	20	23
42	·9544	·9570	·9595	·9621	·9646	·9671	·9697	47	3	5	8	10	13	15	18	20	23
43	·9697	·9722	·9747	·9773	·9798	·9823	·9848	46	3	5	8	10	13	15	18	20	23
44	·9848	·9874	·9899	·9924	·9949	·9975	10·0000	45	3	5	8	10	13	15	18	20	23
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

LOGARITHMIC COTANGENTS.

LOGARITHMIC TANGENTS.

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
45°	10·0000	10·0025	10·0051	10·0076	10·0101	10·0126	10·0152	44°	3	5	8	10	13	15	18	20	23
46	·0152	·0177	·0202	·0228	·0253	·0278	·0303	43	3	5	8	10	13	15	18	20	23
47	·0303	·0329	·0354	·0379	·0405	·0430	·0456	42	3	5	8	10	13	15	18	20	23
48	·0456	·0481	·0506	·0532	·0557	·0583	·0608	41	3	5	8	10	13	15	18	20	23
49	·0608	·0634	·0659	·0685	·0711	·0736	·0762	40°	3	5	8	10	13	15	18	20	23
50°	10·0762	10·0788	10·0813	10·0839	10·0865	10·0890	10·0916	39	3	5	8	10	13	15	18	21	23
51	·0916	·0942	·0968	·0994	·1020	·1046	·1072	38	3	5	8	10	13	16	18	21	23
52	·1072	·1098	·1124	·1150	·1176	·1203	·1229	37	3	5	8	10	13	16	18	21	24
53	·1229	·1255	·1282	·1308	·1334	·1361	·1387	36	3	5	8	11	13	16	19	21	24
54	·1387	·1414	·1441	·1467	·1494	·1521	·1548	35	3	5	8	11	13	16	19	21	24
55	10·1548	10·1575	10·1602	10·1629	10·1656	10·1683	10·1710	34	3	5	8	11	14	16	19	22	24
56	·1710	·1737	·1765	·1792	·1820	·1847	·1875	33	3	5	8	11	14	16	19	22	25
57	·1875	·1903	·1930	·1958	·1986	·2014	·2042	32	3	6	8	11	14	17	20	22	25
58	·2042	·2070	·2098	·2127	·2155	·2184	·2212	31	3	6	9	11	14	17	20	23	26
59	·2212	·2241	·2270	·2299	·2327	·2356	·2386	30°	3	6	9	12	14	17	20	23	26
60°	10·2386	10·2415	10·2444	10·2474	10·2503	10·2533	10·2562	29	3	6	9	12	15	18	21	24	27
61	·2562	·2592	·2622	·2652	·2683	·2713	·2743	28	3	6	9	12	15	18	21	24	27
62	·2743	·2774	·2804	·2835	·2866	·2897	·2928	27	3	6	9	12	15	19	22	25	28
63	·2928	·2960	·2991	·3023	·3054	·3086	·3118	26	3	6	9	13	16	19	22	25	28
64	·3118	·3150	·3183	·3215	·3248	·3280	·3313	25	3	7	10	13	16	20	23	26	29

ANSWERS.

I. (Page 3.)

1. $\frac{2}{3}$. 2. $\frac{301}{360}$. 3. $\frac{45569}{64800}$. 4. $1\frac{9}{20}$.

5. $2\frac{3661}{10800}$. 6. $4\frac{388}{3375}$.

14. $5^\circ 33' 20''$; $66^\circ 40'$. 15. $33^\circ 15' 45''$; $66^\circ 31' 30''$.

16. $74^\circ 5' 26''$; $44^\circ 21' 57''$; $61^\circ 32' 37''$. 17. $138^\circ 4' 58''$.

IV. (Page 19.)

5. $\frac{\sqrt{15}}{4}$; $\frac{1}{\sqrt{15}}$ etc. 6. $\frac{12}{5}$. 7. $\frac{11}{60}$; $\frac{60}{61}$; $\frac{61}{60}$.

8. $\frac{3}{5}$; $\frac{4}{3}$. 9. $\frac{40}{9}$; $\frac{41}{40}$. 10. $\frac{3}{5}$, $\frac{4}{5}$; $\frac{5}{3}$.

11. ·75; 2·676. 12. 2·51; 3·64. 13. $\frac{1}{2}\sqrt{5}$; $\frac{3}{5}\sqrt{5}$.

14. $\frac{1}{2}$. 15. $\frac{3}{5}$ or $\frac{5}{13}$. 16. $\frac{5}{13}$. 17. $\frac{12}{13}$

18. $\frac{1}{\sqrt{3}}$ or 1. 19. $\frac{1}{2}$. 20. 1, $\frac{1}{3}\sqrt{3}$. 21. $1+\sqrt{2}$.

22. $\frac{2x(x+1)}{2x^2+2x+1}$; $\frac{2x+1}{2x^2+2x+1}$.

V. (Pages 26, 27.)

1. $208\frac{1}{3}$ yds. 2. 6000 ft. 3. 462 ft.
4. 346·41 ft. 5. 34·64... ft.; 20 ft.
6. 160 ft. 7. 225 ft. 8. 136·6 ft.
9. 146·4... ft. 10. 367·9... yards; 454·3... yards.
11. 28 ft. 12. 87·846... ft.
13. 94·641... ft.; 54·641... ft. 14. 86·6... ft.
15. 115·359... ft. 16. 43·3... ft.; 75 ft. from one of the pillars.
17. 30°. 19. 13·8564 miles per hour.

VI. (Pages 33, 34.)

1. Approximately 42° and 34°. 2. ·97 nearly.
3. 600 ft. 4. $\frac{2xy}{x^2+y^2}$; $\frac{2xy}{x^2-y^2}$.
8. $\frac{1}{\tan^4 A} - \tan^4 A$. 9. $\theta = 60°$.
10. In $1\frac{1}{3}$ minutes.

VII. (Page 42.)

1. $-\cos 25°$. 2. $\sin 6°$. 3. $-\tan 43°$.
4. $\sin 12°$. 5. $-\cos 43°$. 6. $\cot 35°$.
7. Negative. 8. Positive. 9. Negative.
10. Negative. 11. Negative. 12. Positive.
13. Negative. 14. Negative. 15. $-\frac{\sqrt{3}-1}{2}$; $\frac{4}{3}\sqrt{3}$.
16. $-\frac{\sqrt{3}+1}{2} - \frac{4}{3}\sqrt{3}$. 17. $-\sqrt{2}$; -2. 18. $\sqrt{2}$; -2.
19. $-\frac{\sqrt{3}-1}{2}$; $\frac{4}{3}\sqrt{3}$.

VIII. (Page 47.)

1. $\frac{77}{85}$. 2. $\frac{4}{5}$. 3. $\frac{16}{65}$; $\frac{33}{65}$.
4. $-\frac{133}{205}$; $-\frac{84}{205}$. 5. $\frac{1596}{3445}$; $\frac{3444}{3445}$.
6. $\frac{220}{221}$; $\frac{171}{221}$; $\frac{220}{21}$.

X. (Pages 53, 54.)

1. $\sin 5\theta + \sin\theta$.
2. $\sin 4\theta - \sin 2\theta$.
3. $\cos 7\theta + \cos 3\theta$.
4. $\sin 11\theta - \sin 7\theta$.
5. $\cos\theta - \cos 3\theta$.
6. $\cos 2\theta - \cos 12\theta$.
7. $\sin 12\theta - \sin 2\theta$.
8. $\cos 14\theta + \cos 8\theta$.
9. $\cos 12^\circ - \cos 120^\circ$.
10. $\sin 5\theta + \sin 2\theta$.
11. $\sin 4\theta - \sin\theta$.
12. $\cos 2\theta + \cos\frac{3\theta}{2}$.
13. $\cos\theta - \cos\frac{3\theta}{2}$.
26. $\cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta$.

XI. (Pages 57, 58.)

1. -1; $\frac{1}{7}$.
2. $-\frac{117}{44}$; $\frac{3}{4}$.
3. $\frac{84}{13}$.
5. $\frac{4}{3}$; $\frac{3}{4}$; 3; $\frac{9}{13}$.
12. 1.

XII. (Pages 61—63.)

1. (1) $\pm\frac{24}{25}$; (2) $\pm\frac{120}{169}$; (3) $\frac{2016}{4225}$.
2. (1) $\frac{161}{289}$; (2) $-\frac{7}{25}$; (3) $\frac{119}{169}$.
3. a.
31. $\cot\beta$ and $-\tan\beta$.
32. $\cos a$ and $\sin a$.

XIII. (Pages 67, 68.)

1. $\frac{2\sqrt{2}+\sqrt{3}}{6}$; $\frac{7\sqrt{3}+4\sqrt{2}}{18}$.
2. $\frac{13}{12}$; $\frac{1}{2}\sqrt{13}$; $\frac{169}{120}$.
3. $\frac{7}{5\sqrt{2}}$.
4. $\frac{16}{305}$; $\frac{49}{305}$.
5. $\pm\frac{3}{4}$; $\pm\frac{1}{3}$.
6. $\pm\frac{3}{4}$.
7. $\sqrt{2}-1$; $-(\sqrt{2}+1)+\sqrt{4+2\sqrt{2}}$.
9. $\frac{a^2+b^2}{2}-1$; $\sqrt{\frac{4}{a^2+b^2}-1}$.

XV. (Pages 80, 81.)

1. $\bar{1}\cdot90309$; $\bar{3}\cdot4771213$; $\bar{2}\cdot0334239$; $\bar{1}\cdot4650389$.
2. $\cdot1553361$; $2\cdot1241781$; $\cdot5388340$; $\bar{1}\cdot0759623$.
3. 2; $\bar{2}$; 0; $\bar{4}$; $\bar{2}$; 0; 3.
4. ·3129.
5. 1·3205; 5·8845; ·4618.
6. (1) 21; (2) 13; (3) 30; (4) the 7th; (5) the 21st; (6) the 32nd.
7. ·2222.
8. 8·6415.
9. 6·0693.
10. 1·6389.
11. 4·717.
12. ·41431.
13. 6·736.
14. 1328·7.
15. 333050 cubic centimetres.

XVI. (Page 86.)

1. 4·5527375; 1·5527394.
2. 4·7689529; $\bar{3}\cdot7689502$.
3. 478·475; ·004784777.
4. 2·583674; ·0258362.
5. 1·2816.
6. 1·1325.
7. 51·125.
8. 1·3146.
9. 217·95.
10. ·58575.
11. 225·7.

XVII. (Pages 91, 92).

1. ·68704.
2. 43° 23′ 43″.
3. ·31333; 18° 15′ 28′.
4. ·84551; ·84545.
5. 32° 16′ 36″; 32° 16′ 20′.
6. 4·12031; 4·12188.
7. 4·39932; 4·39768.
8. 13° 8′ 47″.
9. 9·74148; 33° 27′ 25″.
10. 9·9147334.
11. 34° 44′ 27″.
12. 9·5254497; 71° 27′ 43″.
13. 10·0229414.
14. 18° 27′ 17″.
15. 36° 52′ 12′.

XVIII. (Pages 95, 96.)

1. ·61245.
2. 3·37235.
3. ·41284.
4. ·47392.
5. 1·71406.
6. ·30082.
7. 9·42568.
8. 9·90082.
9. 9·67433.
10. 13° 28′.
11. 22° 1′.
12. 65° 24′.
13. 22° 37′.
14. 10° 15′.
15. 44° 56′.
16. 22° 37′ 34″.
17. 60° 42′.
18. 79° 17′ 34″.
19. 73° 31′ 13″.
20. 14° 3′ 8″.
21. 64° 51′ 7″.
22. 8° 9′ 33″.
23. 50° 27′.
24. 73° 30′.

XIX. (Page 103.)

1. $\frac{1}{5}$, $\frac{1}{2}$, and $\frac{9}{7}$.

2. $\frac{4}{\sqrt{41}}$, $\frac{3}{5}$, and $\frac{8}{5\sqrt{41}}$; $\frac{40}{41}$, $\frac{24}{25}$, and $\frac{496}{1025}$.

3. $\frac{3}{5}$, $\frac{4}{5}$, and 1.

4. $\frac{5}{12}$, $\frac{12}{5}$, and ∞.

5. $\frac{4}{5}$, $\frac{56}{65}$ and $\frac{12}{13}$.

6. $\frac{7}{41}$ and $\frac{287}{816}$.

7. $-\frac{4}{5}$; 143° 8′.

8. 26° 23′; 36° 20′; 117° 17′.

XX. (Pages 107, 108.)

20. 16·8 feet.

XXII. (Pages 114, 115.

1. 186·60... and 193·18.
2. 26° 33′ 54″; 63° 26′ 6″; $10\sqrt{5}$ ft.
3. 26° 13′ 36″; 734·3.
4. 9° 36′.
5. 17·365 yds.
6. 199·056 feet.
7. 48° 35′ 25″, 36° 52′ 12″ and 94° 32′ 23″.
8. 75° and 15°.

XXIII. (Pages 117, 118.)

1. 90°.
2. 30°.
4. 120°.
5. 45°, 120° and 15°.
6. 45°, 60°, and 75°.
7. 58° 59′ 34″.
8. 104° 28′ 39″.
9. 77° 19′ 9″.
10. 76° 39′ 5″.
11. 44° 25′, 34° 3′, 101° 32′.
12. 25° 3′; 28° 4′; 126° 53′.
13. 56° 15′, 59° 51′ and 63° 54′.
14. 38° 57′, 47° 41′ and 93° 22′.
15. 130° 42′, 23° 27′, and 25° 51′.
16. 38° 18′; 96° 39′; 45° 3′.
17. 41° 7′.

XXIV. (Pages 122, 123.)

1. $A=45°$; $B=75°$; $c=\sqrt{6}$.
2. 40 yds.; 120°; 30°.
3. $\sqrt{6}$; 15°; 105°.
4. ·8965.
6. 87° 27′ 25″; 32° 32′ 35″.
7. 117° 38′ 45″; 27° 38′ 45″.
8. 63° 13′ 1″; 43° 58′ 29″.
9. $8\sqrt{7}$ feet; 79° 6′ 23″; 60°; 40° 53′ 37″.
10. 40° 53′ 36″; 19° 6′ 24″; $\sqrt{7}:2$.
11. 71° 44′ 30″; 48° 15′ 30″.
12. 124° 48′ 59″; 33° 11′ 1″.
13. 102° 22′ 6″; 40° 37′ 54″.
14. $B=38° 30'$; $C=97° 30'$; $a=30{\cdot}13$.
15. 124° 29′ 20″; 21° 54′ 40″.
16. $A=83° 8'$; $B=42° 16'$; $c=199{\cdot}1$.
17. $B=110° 48'$; $C=26° 56'$; $a=93{\cdot}55$.
18. 73° 0′ 37″; 48° 41′ 23″.
19. 88° 30′ 27″; 33° 31′ 33″.

XXV. (Pages 127, 128.)

1. There is no triangle.
2. $B_1=30°$, $C_1=105°$, and $b_1=\sqrt{2}$; $B_2=60°$, $C_2=75°$, and $b_2=\sqrt{6}$.
3. $B_1=30°$, $C_1=120°$, and $b_1=100$; $B_2=90°$, $C_2=60°$, and $b_2=200$.
5. $4\sqrt{3}\pm 2\sqrt{5}$.
7. 33° 29′ 30″ and 101° 30′ 30″.
8. (1) The triangle is right-angled and $B=60°$.
   (2) $b_1=60{\cdot}388$, $B_1=8° 41'$ and $C_1=141° 19'$;
   $B_2=111° 19'$ and $C_2=38° 41'$.
9. $c_1=17{\cdot}1$, $C_1=108° 11' 21''$; $c_2=3{\cdot}68$, $C_2=11° 48' 39''$.
10. 5·988... and 2·6718... miles per hour.
11. $C_1=88° 57'$, $c_1=7{\cdot}9987$; $C_2=31° 3'$, $c_2=4{\cdot}1263$.
12. $C=12° 22'$; $A=137° 38'$; $a=9{\cdot}434$.
13. $B_1=116° 50'$, $b_1=13{\cdot}047$; $B_2=23° 10'$, $b_2=5{\cdot}751$.
14. 63° 2′ 12″ or 116° 57′ 48″.
15. 62° 32′ 33″ and 102° 16′ 27″, or 117° 27′ 27″ and 47° 21′ 33″.
16. 5926·7.

XXVI. (Pages 129, 130.)

1. $7:9:11$. 4. $79 \cdot 07$.
5. 1 mile; 1·219714... miles. 7. 20·976... ft.
8. 6·8567... and 5·4378... ft. 9. 404·43... ft.
10. 500·4 feet. 11. $\sqrt{3}-1$, 1, and $\sqrt{2}$ feet.
12. 233·25 yards. 13. 2229 yards.

XXVII. (Pages 137–141.)

1. 100 ft. high and 50 ft. broad; 25 feet. 2. 25·78 yds.
3. 10·2426... miles per hour. 4. 18·3... ft.
5. 120 ft. 6. 1939·2... ft.
7. 10 miles per hour. 8. 16·3923... miles; 14·697... miles.
9. 114·4123 ft. 10. 86·6... yards.
11. $c \sin\beta \operatorname{cosec}(\alpha+\beta)$; $c \sin\alpha \sin\beta \operatorname{cosec}(\alpha+\beta)$.
12. $32\sqrt{5}=71 \cdot 55...$ ft. 13. 100 ft.
15. $PQ=BP=BQ=1000$ ft.; $AP=500(\sqrt{6}-\sqrt{2})$ ft.; $AQ=1000\sqrt{2}$ ft.
16. 1·366 miles. 18. 1069·8 ft.
19. 25·98... ft.; 70·98... ft.; 85·98... ft. 21. 692·8... yards.
22. $h \tan\alpha \cot\beta$. 23. 61·224... ft.
26. $l \operatorname{cosec}\gamma$, where γ is the sun's altitude; $\sin\gamma=\frac{2}{7}$.
28. At a distance $\frac{375}{\sqrt{7}}$ ft. from the cliff.

XXVIII. (Pages 141–143.)

1. 51·77 yds.; 11° 18′ 36″. 2. 104·9 feet.
3. ·32129 miles. 4. ·17365 miles; ·98481 miles.
5. 141·346 and 41·92 yds. 6. 1961 yds. nearly.
7. 4·7403 and 5·1508 miles. 8. 1·8566 and 1·2471 miles.
9. 624·76 yards. 10. 91·896 ft. 11. 2·4583 miles.
12. 439·2 yds. 13. 132·27 feet. 14. 235·8 yds.
15. 1·4277 miles. 16. 108·69 feet. 17. 119·286 ft.
18. 401 ft.; 377 ft.; E. 33° 41′ N. 19. 125·32 feet.

XXIX. (Pages 145, 146.)

1. 84. **2.** 216. **3.** 630. **4.** 3720. **5.** 270.

9. 11·684 sq. ins. **10.** 155333 sq. ft. = 3·565... acres.

11. 60499 sq. ft. = 1·39 acres. **12.** 481·6 sq. ft.

13. 85306 sq. ft. **14.** 6464·1 sq. yds.

15. 16479·1 sq. yds. = 3·405 acres. **17.** 5 and 3·18 ins.

18. 90 and 69·65 feet. **19.** 35 yds. and 26 yds.

20. 14·941... inches. **21.** 120°.

XXX. (Pages 151, 152.)

3. $8\frac{1}{3}$, $1\frac{1}{2}$, 8, 2, and 24 respectively.

XXXI. (Page 156.)

1. 77·98 ins. **2.** ·5359.

3. (1) 1·720... sq. ft.; (2) 2·598... sq. ft.;
(3) 4·8284... sq. ft.; (4) 7·694... sq. ft.;
(5) 11·196... sq. ft.

4. 1·8866... sq. ft. **5.** 3·3136... sq. ft.

6. $2+\sqrt{2}:4$; $\sqrt{2+\sqrt{2}}:2$. **10.** 6.

XXXII. (Page 159.)

1. $33^g\,33^{`}\,33\cdot\dot{3}^{``}$. **2.** $90^g$. **3.** $153^g\,88^{`}\,88\cdot\dot{8}^{``}$.

4. $39^g\,76^{`}\,38\cdot\dot{8}^{``}$. **5.** $261\,34^{`}\,44\cdot\dot{4}^{``}$. **6.** $528^g\,3^{`}\,33\cdot3^{``}$.

7. $1\frac{1}{5}$ rt. ∠; 108°. **8.** ·453524 rt. ∠; 40° 49′ 1·776″.

9. ·394536 rt. ∠; 35° 30′ 29·664″.

10. 2·550809 rt. ∠; 229° 34′ 22·116′.

11. 7·590005 rt. ∠; 683° 6′ 1·62″. **15.** $47\frac{7}{10}$°; $42\frac{13}{16}$°.

17. 83° 20′; 10° 48′.

XXXIII. (Page 163.)

1. 25132·74 miles nearly.
2. 19·28 miles per hour nearly.
3. 12·85 miles nearly.
4. 3·14159... inches.
5. 581,194,640 miles nearly.
6. 7·8539... feet.
7. 720·29....
8. 14·994 miles nearly.

XXXIV. (Pages 166, 167.)

1. 60°. 2. 240°. 3. 1800°. 4. 57° 17′ 44·8″.
5. 458° 21′ 58·4″. 6. 160^g. 7. $233^g\, 33^{\backprime}\, 33\cdot\dot{3}^{\backprime\backprime}$.
8. 2000^g. 9. $\frac{\pi}{3}$. 10. $\frac{221}{360}\pi$. 11. $\frac{703}{720}\pi$.
12. $\frac{3557}{13500}\pi$. 13. $\frac{79}{36}\pi$. 14. $\frac{3\pi}{10}$. 15. $\frac{1103}{2000}\pi$.
16. $1\cdot726268\pi$. 17. 81°; 9°. 18. 132° 15′ $12\cdot\dot{6}''$.
19. 30°, 60°, and 90°. 20. $\frac{1}{2}$, $\frac{\pi}{3}$, and $\frac{2\pi}{3}-\frac{1}{2}$ radians.
21. (1) $\frac{3\pi}{5}$; 108°. (2) $\frac{5\pi}{7}$; $128\frac{4}{7}°$. (3) $\frac{3\pi}{4}$; 135°.
(4) $\frac{5\pi}{6}$; 150°. (5) $\frac{15\pi}{17}$; $158\frac{14}{17}°$.
22. 1 : 15 : 360. 23. 60°; 45°; 135°; 120°.
24. 14; $\frac{6\pi}{7}$. 25. 8 and 4. 26. $\frac{\pi}{3}$.
27. (1) $\frac{5\pi^c}{12}=75°=83\frac{1}{3}^g$; (2) $\frac{7\pi^c}{18}=70°=77\frac{7}{9}^g$;
(3) $\frac{5\pi^c}{8}=112\frac{1}{2}°=125^g$.
28. (1) At $7\frac{7}{11}$ and 36 minutes past 4; (2) at $28\frac{4}{11}$ and 48 minutes past 7.

XXXV. (Pages 170, 171.)

1. 20·455° nearly.
2. $\frac{3}{5}$ radian; 34° 22′ 38·9″.
3. 68·75 inches nearly.
4. ·05236 inch nearly.
5. 24·555 inches nearly.
6. 1° 25′ 57″ nearly.

7. 3959·8 miles nearly.
8. π ft. $= 3\cdot14159$ ft.
9. 5 : 4.
10. 3·1416.
11. $\frac{4\pi}{35}$, $\frac{9\pi}{35}$, $\frac{14\pi}{35}$, $\frac{19\pi}{35}$, and $\frac{24\pi}{35}$ radians.
12. 2062·65 ft. nearly.
13. 1·5359 ft. nearly.
14. 262·6 ft. nearly.
15. 32142·9 ft. nearly.
16. 38197·2 ft. nearly.
17. 19·1′ nearly.
18. 1105·8 miles.

XXXVI. (Page 182.)

12. 5; 53° 8′.
13. $\sqrt{2}$; $-45°$.
14. $\sqrt{5}$; $-26° 34'$.

XXXVII. (Pages 188, 189.)

4. 1·4142...; -2.
5. 1·366...; $-2\cdot3094$....
6. 45° and 135°.
7. 120° and 240°.
8. 135° and 315°.
9. 150° and 330°.
10. 150° and 210°.
11. 210° and 330°.
12. sin 17°.
13. $-$ cot 24°.
14. cos 33°.
15. $-$ cos 28°.
16. cot 25°.
17. cos 30°.
18. cot 26°.
19. $-$ cosec 23°.
20. $-$ cosec 36°.
21. negative.
22. positive.
23. zero.
24. positive.
25. positive.
26. positive.
27. negative.
28. $\frac{1}{\sqrt{3}}$ and $\frac{-\sqrt{2}}{\sqrt{3}}$; $\frac{-1}{\sqrt{3}}$ and $\frac{\sqrt{2}}{\sqrt{3}}$.

XXXVIII. (Page 198.)

1. $n\pi + (-1)^n \frac{\pi}{6}$.
2. $n\pi - (-1)^n \frac{\pi}{3}$.
3. $n\pi + (-1)^n \frac{\pi}{4}$.
4. $2n\pi \pm \frac{2\pi}{3}$.
5. $2n\pi \pm \frac{\pi}{6}$.
6. $2n\pi \pm \frac{3\pi}{4}$.
7. $n\pi + \frac{\pi}{3}$.
8. $n\pi + \frac{3\pi}{4}$.
9. $n\pi + \frac{\pi}{4}$.
10. $2n\pi \pm \frac{\pi}{3}$.
11. $n\pi + (-1)^n \frac{\pi}{3}$.
12. $n\pi \pm \frac{\pi}{2}$.
13. $n\pi \pm \frac{\pi}{3}$.
14. $n\pi \pm \frac{\pi}{6}$.
15. $n\pi \pm a$.
16. $n\pi \pm \frac{\pi}{4}$.
17. $n\pi \pm \frac{\pi}{6}$.
18. $(2n+1)\pi + \frac{\pi}{4}$.
19. $2n\pi - \frac{\pi}{6}$.
20. (1) 60° and 120°; (2) 120° and 240°; (3) 30° and 210°.

XXXIX. (Page 200.)

1. $n\pi+(-1)^n\frac{\pi}{6}$.

2. $2n\pi\pm\frac{2\pi}{3}$.

3. $n\pi+(-1)^n\frac{\pi}{3}$.

4. $\cos\theta=\frac{\sqrt{5}-1}{2}$.

5. $n\pi+(-1)^n\frac{\pi}{10}$ or $n\pi-(-1)^n\frac{3\pi}{10}$ (Arts. 68, 69).

6. $\theta=2n\pi\pm\frac{\pi}{3}$.

7. $\theta=n\pi+\frac{\pi}{4}$ or $n\pi+\frac{\pi}{3}$.

8. $\theta=n\pi+\frac{2\pi}{3}$ or $n\pi+\frac{5\pi}{6}$.

9. $\tan\theta=\frac{1}{a}$ or $-\frac{1}{b}$.

10. $\theta=n\pi\pm\frac{\pi}{4}$.

11. $\theta=2n\pi$ or $2n\pi+\frac{\pi}{4}$.

12. $n\pi\pm\frac{\pi}{6}$.

13. $n\pi$ or $2n\pi\pm\frac{\pi}{3}$.

14. $2n\pi\pm\frac{\pi}{3}$ or $2n\pi\pm\frac{\pi}{6}$.

15. $\sin\theta=1$ or $-\frac{1}{3}$.

16. $\frac{n\pi}{5}+(-1)^n\frac{\pi}{20}$.

17. $\frac{n\pi}{4}$ or $\frac{(2n+1)\pi}{10}$.

18. $2n\pi$ or $\frac{(2n+1)\pi}{5}$.

19. $\frac{2r\pi}{m-n}$ or $\frac{2r\pi}{m+n}$.

20. $\left(2n+\frac{1}{2}\right)\frac{\pi}{5}$ or $2n\pi-\frac{\pi}{2}$.

21. $2n\pi$ or $\frac{2n\pi}{9}$.

22. $\left(2r+\frac{1}{2}\right)\frac{\pi}{m+n}$ or $\left(2r-\frac{1}{2}\right)\frac{\pi}{m-n}$.

23. $\left(n+\frac{1}{2}\right)\frac{\pi}{9}$.

24. $\left(m+\frac{1}{2}\right)\frac{\pi}{n+1}$.

25. $\frac{n\pi}{4}\pm\sqrt{1+\frac{n^2\pi^2}{16}}$.

26. $\left(n+\frac{1}{2}\right)\frac{\pi}{3}$.

27. $\left(n+\frac{1}{2}\right)\frac{\pi}{3}\pm\frac{a}{3}$.

28. $\left(n+\frac{1}{2}\right)\frac{\pi}{4}$.

29. $\frac{n\pi}{3}\pm\frac{a}{3}$.

30. $n\pi\pm\frac{\pi}{6}$.

31. $\left(r+\frac{1}{2}\right)\frac{\pi}{m-n}$.

32. $\theta=\left(m+\frac{n}{2}\right)\pi\pm\frac{\pi}{6}+(-1)^n\frac{\pi}{12}$; $\phi=\left(m-\frac{n}{2}\right)\pi\pm\frac{\pi}{6}-(-1)^n\frac{\pi}{12}$.

33. $\frac{1}{5}\left[(6m-4n)\pi\pm\frac{\pi}{2}\mp\frac{2\pi}{3}\right]$; $\frac{1}{5}\left[(6n-4m)\pi\pm\pi\mp\frac{\pi}{3}\right]$.

34. $\frac{1}{3}$ or $\frac{5}{3}$.

XL. (Pages 203, 204.)

1. $\frac{n\pi}{4}$ or $\frac{1}{3}\left(2n\pi \pm \frac{\pi}{3}\right)$.

2. $\left(n+\frac{1}{2}\right)\frac{\pi}{4}$ or $\left(2n \pm \frac{1}{3}\right)\frac{\pi}{3}$.

3. $\left(n+\frac{1}{2}\right)\frac{\pi}{2}$ or $2n\pi$.

4. $\left(n+\frac{1}{2}\right)\frac{\pi}{3}$ or $n\pi+(-1)^n\frac{\pi}{6}$.

5. $\frac{2n\pi}{3}$ or $\left(n+\frac{1}{4}\right)\pi$ or $\left(2n-\frac{1}{2}\right)\pi$.

6. $\frac{n\pi}{3}$ or $\left(2n \pm \frac{1}{3}\right)\frac{\pi}{4}$.

7. $\left(n+\frac{1}{2}\right)\frac{\pi}{2}$ or $2n\pi \pm \frac{2\pi}{3}$.

8. $n\frac{\pi}{3}$ or $\left(n \pm \frac{1}{3}\right)\pi$.

9. $2n\pi$ or $\left(\frac{2n}{3}+\frac{1}{2}\right)\pi$.

10. $n\pi+(-1)^n\frac{\pi}{6}$ or $n\pi+(-1)^n\frac{\pi}{10}$ or $n\pi-(-1)^n\frac{3\pi}{10}$.

11. $\left(n+\frac{1}{2}\right)\frac{\pi}{8}$ or $\left(n+\frac{1}{2}\right)\frac{\pi}{2}$.

12. $\frac{n\pi}{8}$ or $\frac{n\pi}{4} \pm \frac{\pi}{24}$.

13. $\frac{2n\pi}{7} \pm \frac{\pi}{14}$ or $n\pi \pm \frac{\pi}{4}$.

14. $\frac{2r\pi}{m+n}$ or $(2r+1)\frac{\pi}{m-n}$.

15. $(2r+1)\frac{\pi}{m \pm n}$.

16. $m\pi$ or $\frac{m\pi}{n-1}$ or $\left(m+\frac{1}{2}\right)\frac{\pi}{n}$.

17. $2n\pi-\frac{\pi}{2}$; $\frac{1}{5}\left(2n\pi-\frac{\pi}{2}\right)$.

18. $n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{3}$.

19. $2n\pi+\frac{\pi}{4}$.

20. $n\pi+\frac{\pi}{6}+(-1)^n\frac{\pi}{4}$.

21. $2n\pi+\frac{\pi}{4} \pm A$.

22. $-21°48'+n\,.\,180°+(-1)^n\,[68°12']$.

23. $2n\,.\,180°+78°58'$; $2n\,.\,180°+27°18'$.

24. $n\,.\,180°+45°$; $n\,.\,180°+26°34'$.

25. $2n\pi$ or $2n\pi+\frac{2\pi}{3}$.

26. $2n\pi$ or $2n\pi+\frac{\pi}{2}$.

27. $2n\pi+\frac{\pi}{2}$ or $2n\pi-\frac{\pi}{3}$.

28. $2n\pi+\frac{\pi}{6}$.

29. $n\pi$.

30. $\sin\theta=\frac{\pm\sqrt{17}-1}{8}$.

31. $\cos\theta=\frac{\sqrt{17}-3}{4}$.

32. $n\pi\pm\frac{\pi}{3}$ or $n\pi+\frac{\pi}{2}$.

33. $\left(n+\frac{1}{4}\right)\frac{\pi}{2}$.

34. $n\pi\pm\frac{\pi}{4}$.

35. $n\pi+\frac{\pi}{4}$.

36. $\theta=\frac{n\pi}{2}$ or $n\pi\pm\frac{\pi}{3}$; also $\theta=n\pi\pm\frac{a}{2}$, where $\cos a=\frac{1}{3}$.

37. $\left(n+\frac{1}{3}\right)\frac{\pi}{3}$.

38. $n\pi\pm\frac{\pi}{3}$.

XLI. (Pages 209, 210.)

14. $\pm\sqrt{\sin 2\beta}$.

15. $\frac{1}{6}$.

16. $\pm\frac{1}{\sqrt{2}}$.

17. $\frac{4}{7}\sqrt{21}$.

18. $\frac{1}{4}$.

19. $\frac{\sqrt{5}}{3}$.

20. $\sqrt{3}$ or $-(2+\sqrt{3})$.

21. $\sqrt{3}$, or $2-\sqrt{3}$.

22. $\frac{1}{14}\sqrt{21}$.

23. 13.

XLII. (Page 214.)

1. ·00204.

2. ·00007.

3. ·00029.

4. ·99999.

5. 25783·10077.

6. 1·0000011.

7. $34'23''$.

8. $28°40'37''$.

9. $39'42''$.

10. $2°33'44''$.

11. 114·59... inches.

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(Page 218) LIII.

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(Page 218) LIV.

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